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THE

PHYSICAL REVIEW.

THE GENERAL PRINCIPLE OF RELATIVITY APPLIED TO THE RUTHERFORD-BOHR ATOM-MODEL.

By Th. Wereide.

Abstract.

Application of the general principle of relativity to the Rutherford-Bohr atom.—In this atom the possible electron orbits are determined according to mechanical principles, then the actual orbits are selected by the quantum conditions. In computing the possible orbits, Bohr assumed classical mechanics, Sommerfeld applied the special theory of relativity with marked success, and now his assumptions are further generalized. In the expression for the world line element ds in polar coördinates, the potential functions γ and γ' are put equal to $(1 + a\kappa/r + b\kappa^2/r^2)$ and $(1 + a'\kappa/r)$ respectively, where κ is the ratio of potential energy to mass. The quantum assumptions are the same as Sommerfeld's, with generalized ds, with an additional alternative assumption that the radial expression involves γ'. These lead to expressions for the energy in which all the constants are known from experimental results, except B which Sommerfeld puts equal to $1/4$ but which has not yet been experimentally determined. The method here used for the atom is the same as Eddington used for a macrocosmos.

IN his foundation of the modern theory of atomic structures Bohr makes use of two principles. The first is the principle of *classical mechanics*. The electron has a certain mass and moves under the influence of a force of given intensity. By means of these two quantities, the force and the mass, the *possible* orbits are calculated by applying the laws of classical mechanics. The second principle is the *quantum conditions* by means of which a *selection* is undertaken between the possible orbits calculated from classical mechanics.

Now the general principle of relativity demands that classical mechanics are not exact, a very small, but in certain cases observable, correction being necessary. This correction ought consequently also to be undertaken for the mechanical formulas which are used in calculating the orbits of the electrons in the atom.

The correctness of this assumption has been demonstrated by Sommerfeld. By undertaking a relativistic correction of the kinetic energy he has deduced a theory of the fine structure of spectral lines, and this theory has been verified in a splendid way by Paschen's spectroscopical measurements.[1]

The principle of relativity applied by Sommerfeld, however, is not the general principle but the special, in spite of the circumstance that the electron is moving with variable velocity and in a rather intense field of force. As a first approximation Sommerfeld's relativistic correction is no doubt sufficient, but it would be more satisfying if this approximation could be obtained by applying the *general* principle of relativity. The necessity of applying the principle in its general form appears also from the fact that the classical Coulomb's law must be modified for sufficiently small dimensions, and the nature of this modification ought to be predicted by the general principle of relativity supposing that the exact formulation of this principle can be found for the system considered.

We shall, in the following, consider an electron moving in a radially symmetrical field produced by a nucleus of charge E. In order to apply the general principle of relativity to this electron we must answer the following three questions:

I. What is the exact form of the world line element ds of the electron?
II. What is the exact form of the energy of electron and field?
III. How are the quantum conditions to be formulated?

The following development will show that the solution of these questions, and thus the relativistic problem of the atom, is not so difficult as it might seem at first sight.

I. The World Line Element.

In the relativistic theories it is generally assumed that the world line element ds, in the case of a radially symmetrical field, may be written in the form

$$ds^2 = \gamma du^2 - \gamma' dr^2 - r^2 d\varphi^2, \tag{1}$$

where $u = ct$, r and φ being the polar coördinates of the electron with the center of the field as origin, and γ and γ' being functions of r.

In Einstein's theory the potentials γ and γ' are calculated a priori.

A. S. Eddington [2] has shown that the potentials may also be determined empirically. γ and γ' may be developed as functions of r in the

[1] A. Sommerfeld, Atombau und Spektrallinien, 1921, p. 349.
[2] A. S. Eddington, Space Time and Gravitation, 1920, Note 10, p. 207.

following way

$$(1, 1) \qquad \gamma = 1 + a\frac{\kappa}{r} + b\frac{\kappa^2}{r^2} + \cdots,$$

$$(1, 2) \qquad \gamma' = 1 + a'\frac{\kappa}{r} + \cdots,$$

where

$$\frac{\kappa}{r} = -\frac{1}{c^2} \times \frac{\text{variable potential energy classically calculated}}{\text{mass of particle at rest}}.$$

Substituting these values in the dynamical equations and comparing the results with observations, the unknown coefficients a, b and a' may be determined. By this method Eddington finds for a macrocosmos

$a = -2$ (demanded by Newton's law),
$a' = 2$ (demanded by the observed deviation of light),
and $b = 0$ (demanded by the perihelion movement of Mercury, 42″.9 per century).

We shall here, in applying the principle to the atom, use exactly the same method. We assume the line element of the electron to satisfy the conditions (1), (1, 1) and (1, 2) and shall afterwards from spectroscopical data try to determine the values of a, b and a'. If such a determination is possible, we shall arrive at a general principle of relativity for a microcosmos by the same empirical method as used by Eddington for a macrocosmos.

II. The Energy Function.

The energy function must satisfy three conditions:

1. It must be constant during the movement of the electron in the stationary orbits.

2. For vanishing field and velocity, it must approach the classical energy function.

3. For vanishing field alone, it must approach the energy function of the special principle of relativity.

These conditions are satisfied by the function

$$(2) \qquad E = mc^2\gamma\frac{du}{ds} = W + \text{constant},$$

W being the variable terms of the energy function.

The following calculations are most easily carried out by first considering the function

$$(2, 1) \qquad W' = W\left(1 + \frac{W}{2mc^2}\right) = \frac{1}{2}mc^2\gamma^2\left(\frac{du}{ds}\right)^2 + \text{constant}$$

and afterwards going over to W.

Substituting the values (1, 1) and (1, 2) in (2, 1) we get with sufficient approximation

$$(2,\ 2)\qquad W' = \frac{1}{2}mc^2\left(a\frac{\kappa}{r} + b\frac{\kappa^2}{r^2} + \frac{w_\varphi^2}{c^2} + \frac{w_r^2}{c^2} + \frac{a\kappa}{r}\frac{w_\varphi^2}{c^2} + \int\frac{(a+a')\kappa}{r}\frac{w_r^2}{c^2} + \cdots\right),$$

where

$$w_\varphi = \frac{rd\varphi}{d\sigma}, \qquad w_r = \frac{dr}{d\sigma} \qquad \text{and} \qquad \sigma = \frac{s}{c}.$$

III. The Quantum Conditions.

Equation (1), applied to the motion of an electron, will determine the possible orbits. We now have to discuss the problem of the quantum conditions. The movement of the electron will satisfy the condition

$$r^2\frac{d\varphi}{ds} = \frac{rw_\varphi}{c} = H = \text{constant}.$$

In Sommerfeld's theory the same equation holds with the modification that according to him ds has the same meaning as in the special theory of relativity where $\gamma = \gamma' = 1$. The azimuthal quantum condition in Sommerfeld's theory may be written in the form

$$(3)\qquad \oint mr^2\frac{d\varphi}{d\sigma}d\varphi = \oint mrw_\varphi d\varphi = nh,$$

$d\sigma$ having the same value as in the special theory of relativity. The nearest assumption will therefore be to maintain the condition in the form 3 but with the modification that now $d\sigma$ is defined as in the general principle corresponding to equation (1).

Concerning the radial quantum condition, it has, according to Sommerfeld, the form

$$(4,\ 1)\qquad \oint m\frac{dr}{d\sigma}dr = n'h,$$

$d\sigma$ being defined as in the special principle. At first sight it would therefore seem natural to maintain the condition with the above-mentioned modification of defining $d\sigma$ according to equation (1).

In discussing this problem with Professor Bohr, however, he has kindly informed me that the form (4, 1) for the *radial* quantum condition does not agree well with his principle of correspondence. According to a special calculation undertaken by Dr. Kramers the principle of correspondence will be satisfied if the radial quantum condition is written in the form

$$(4,\ 2)\qquad \oint m\gamma'\frac{dr}{d\sigma}dr = \oint m\left(1 + a'\frac{\kappa}{r} + \cdots\right)\frac{dr}{d\sigma}dr = n'h.$$

In order to give the radial quantum condition as wide a form as possible we shall therefore write it in the form

$$\oint m\left(1 + a''\frac{\kappa}{r} + \cdots\right)\frac{dr}{d\sigma}dr = n'h. \tag{4}$$

In this form both alternatives ((4, 1) and (4, 2)) are comprised, $a'' = 0$ giving the first and $a'' = a'$ giving the second alternative.

Having thus answered the three questions in the above way the rest of the problem is only calculation. Substituting (3) in (2, 2) and then (2, 2) in (4) we find W' as function of the quantum numbers n and n'. By the relation (2, 1) we then finally find W to be of the form

$$W = -\left(\frac{a}{2}\right)^2\left(\frac{E}{e}\right)^2\frac{Nh}{(n+n')^2}\left[1 + \left(\frac{E}{e}\right)^2\frac{A\alpha^2}{(n+n')^2}\left(\frac{n'}{n}\right) + \left(\frac{E}{e}\right)^2\frac{B\alpha^2}{(n+n')^2} + \cdots\right],$$

where

$$N = \frac{2\pi^2me^4}{h^3}, \qquad \alpha = \frac{2\pi e^2}{hc},$$

$$A = \tfrac{1}{2}(2a^2 - 2aa'' + aa' - 2b)$$

and

$$B = \tfrac{9}{16}a^2 - b.$$

In order to satisfy the empirical formula of line spectra we must first have $a = -2$.[1]

According to the experimental investigations of Paschen [2] concerning the fine structure of spectral lines we further have

$$A = 1 \text{ (Sommerfeld's value)}.$$

Concerning the coefficient B, for which Sommerfeld's theory demands the value $\frac{1}{4}$, the exact value has not yet been experimentally fixed. It is to be expected, however, that further improvements in experimental spectroscopy will make it possible to determine the constant B also. It will then be possible to determine a' and b and hence the potentials γ and γ'. For the coefficient a'' there seems to be only the two possibilities $a'' = a'$ and $a'' = 0$. We thus get the following two solutions of our problem:

$$\text{I.} \begin{cases} a'' = a', \\ a' = -\frac{3}{4} - B, \\ b = a' + 3, \end{cases} \qquad \text{II.} \begin{cases} a'' = 0, \\ a' = \frac{3}{4} + B, \\ b = -a' + 3, \end{cases}$$

from which a' and b and hence γ and γ' may be calculated as soon as the factor B has been experimentally fixed.

[1] $a = 2$ would give a negative field.

[2] Loc. cit., p. 349.

The above method contains a minimum of arbitrary assumptions, these being essentially limited to the quantum conditions, for which there only seems to be the two alternatives mentioned.

One single further assumption would serve for a complete theoretical determination of the time space of a microcosmos. If we suppose the condition

$$\gamma\gamma' = 1$$

to hold we get $B = -\frac{11}{4}$ and $B = \frac{5}{4}$ respectively. We should then have the following kinds of time spaces corresponding to the empty space round a particle.

Nature of Cosmos.	$\gamma = \frac{1}{\gamma'}$.	Perihelion Movement $\frac{\Delta\varphi}{\varphi}$.
Macrocosmos Einstein's law	$1 - 2\frac{\kappa}{r}$	$3\frac{\kappa^2}{H^2}$
Microcosmos Alternative I	$1 - 2\frac{\kappa}{r} + 5\frac{\kappa^2}{r^2}$	$\frac{1}{2}\frac{\kappa^2}{H^2}$
Microcosmos Alternative II	$\left(1 - \frac{\kappa}{r}\right)^2$	$\frac{5}{2}\frac{\kappa^2}{H^2}$

For the microcosmos of the atom the value of κ is

$$\kappa = \frac{Ee}{mc^2} = \left(\frac{E}{e}\right) 2,7 \cdot 10^{-13} \text{ cm.}$$

Approaching the central charge E at a distance of this order of magnitude the deviation from Coulomb's law will be greater and greater. There seems to be a striking agreement between this value and Rutherford's results concerning the dimensions of atomic nuclei.

CHRISTIANIA,
PHYSICAL LABORATORY OF THE UNIVERSITY,
October 2, 1922.[1]

[1] Paper read at the Second Scandinavian Congress for Physical Research in Upsala, August, 1922. In reading the paper the case $a'' = a'$ was not treated. This has been done afterwards and is due to special investigations by Professor Bohr and Dr. Kramers.

NOTE ON RELATIVISTIC RÖNTGEN L-DOUBLETS AND THE "SCREENING CONSTANT."

By J. B. Green.

Abstract.

Screening constant in the relativity formula for x-ray L-doublet separations has been computed from the wave-lengths for elements of atomic number 74 to 92, in two ways, (1) using the complete formula, and (2) using only the first three terms (Sommerfeld's formula). The second calculation is found to give values more nearly independent of the atomic number than the complete formula, and is therefore to be preferred. This result is shown to be in agreement with a recent *development of Bohr's correspondence principle*, a suggestion by him that in such a series no term should be included which is smaller than the energy radiated by an electron per revolution, as calculated on the classical theory. The wave-length measurements, however, are not yet accurate enough to decide the question definitely.

IN a recent paper by Sommerfeld and Heisenberg,[1] the problem of the sharpness of spectral lines was considered from the standpoint of Bohr's latest extension of the "Correspondence Principle." According to this idea, expressed in his lectures at Göttingen during the past summer, any quantum calculation which involves energies smaller than the energy lost per revolution by an electron revolving about the nucleus, as calculated on the classical basis, should not be taken into account.

In the above-mentioned paper, the authors investigated the question of how many of the relativity corrections to use in the calculation of the L-doublets, without interfering with this point of view. It was suggested by Professor Sommerfeld that the author seek for some evidence in the heaviest atoms that the calculation of the L-doublets is really spoiled by using more than the number of terms which Bohr's point of view would indicate. This was effected in the following manner.

I. The screening constant.—In the theory of the x-ray L-doublets[2] the wave-number difference of the doublet is expressed completely by the formula

$$\frac{\Delta\nu}{R} = \frac{2}{\alpha^2}\left\{\left[1 + \frac{\alpha^2(Z-s)^2}{(n_2' + \sqrt{K_2^2 - \alpha^2(Z-s)^2})^2}\right]^{-1/2} - \left[1 + \frac{\alpha^2(Z-s)^2}{(n_1' + \sqrt{K_1^2 - \alpha^2(Z-s)^2})^2}\right]^{-1/2}\right\}, \quad (a)$$

[1] Sommerfeld and Heisenberg, Zeit. fur Physik, 10, p. 393, 1922.

[2] Sommerfeld, Atombau und Spektrallinien, p. 605.

where $\alpha = 2\pi e^2/ch$, the constant of fine structure; Z is the atomic number, s is the screening constant, R is the Rydberg constant, and n' and K are the radial and azimuthal quantum numbers, respectively. For the L-doublets, $n_2' = K_2 = 1$, $n_1' = 0$, and $K_1 = 2$, so (a) becomes

$$\frac{\Delta\nu}{R} = \frac{1}{\alpha^2}\left[\sqrt{4 - \alpha^2(Z-s)^2} - \sqrt{2}\sqrt{1 + \sqrt{1 - \alpha^2(Z-s)^2}}\right]. \qquad (b)$$

If we expand (a) into a power series in $\alpha^2(Z-s)^2$, we get, indicating each half of (a) by the symbol (n', K),

$$\begin{aligned}(n', K) = (Z-s)^2\Bigg[&\frac{1}{(n'+K)^2} + \frac{\alpha^2(Z-s)^2}{(n'+K)^4}\left\{\frac{1}{4} + \frac{n'}{K}\right\} \\ &+ \frac{\alpha^4(Z-s)^4}{(n'+K)^6}\left\{\frac{1}{8} + \frac{3}{4}\frac{n'}{K} + \frac{3}{2}\left(\frac{n'}{K}\right)^2 + \frac{1}{4}\left(\frac{n'}{K}\right)^3\right\} \\ &+ \frac{\alpha^6(Z-s)^6}{(n'+K)^8}\left\{\frac{5}{64} + \frac{5}{8}\left(\frac{n'}{K}\right) + 2\left(\frac{n'}{K}\right)^2\right. \\ &\qquad \left. + \frac{23}{8}\left(\frac{n'}{K}\right)^3 + \left(\frac{n'}{K}\right)^4 + \frac{1}{8}\left(\frac{n'}{K}\right)^5\right\} \\ &+ \frac{\alpha^8(Z-s)^8}{(n'+K)^{10}}\left\{\frac{7}{128} + \frac{35}{64}\frac{n'}{K} + \frac{75}{32}\left(\frac{n'}{K}\right)^2 + \frac{345}{64}\left(\frac{n'}{K}\right)^3\right. \\ &\left. + \frac{105}{16}\left(\frac{n'}{K}\right)^4 + \frac{207}{64}\left(\frac{n'}{K}\right)^5 + \frac{25}{32}\left(\frac{n'}{K}\right)^6 + \frac{5}{64}\left(\frac{n'}{K}\right)^7\right\} + \cdots\Bigg].\end{aligned} \qquad (c)$$

The method of expansion (and the first three terms) is indicated in "Atombau und Spektrallinien," p. 582. For the L-doublet, $\Delta\nu/R$ then reduces to [3]

$$\frac{\Delta\nu}{R} = \frac{\alpha^2}{2^4}(Z-s)^4\left\{1 + \frac{5}{2}\frac{\alpha^2}{2^2}Z-s)^2 + \frac{53}{8}\frac{\alpha^4}{2^4}(Z-s)^4 + \frac{303}{16}\frac{\alpha^6}{2^6}(Z-s)^6 + \cdots\right\}. \qquad (d)$$

If we take the first three terms of this series, and solve for $(Z-s)^2$, we get

$$(Z-s)^2 = \left(\frac{2^2}{\alpha}\sqrt{\frac{\Delta\nu}{R}} - 5\frac{\Delta\nu}{R}\right)\left(1 + \frac{19}{32}\alpha^2\frac{\Delta\nu}{R}\right). \qquad (e)$$

The "screening constant" s is rather an empirical than a theoretical constant and must be determined from experimental data for $\Delta\nu/R$. It expresses the screening effect of the electrons in the K-shell and the remaining electrons in the L-shell, on an electron tumbling into the L-shell. That is, $(Z-s)$ tells us how much of the total nuclear charge is effective in radiation. For all of the heavier elements, the L-shell is

[3] Page, Bull. Nat. Res. Council, **2**, p. 380, gets 831/32 for the coefficient of the fourth term. This seems to be due to an error in computation.

complete, and since the electrons in the other shells, the M, N, O shells, etc., are far away, and have little or no influence in determining s, we should expect s to be a constant for all of the heavier elements.

The question then arises, shall we use the complete formula (b), or the formula (e), in determining s? Bohr's development of the "Correspondence Principle" would tell us that we should use as many terms in the expansion (d) as will give us energy differences greater than the radiation damping. Duane and Patterson[1] have concluded from their calculations that it would be necessary to use five or six terms in the expansion in order to get a value of s giving $\Delta\nu/R$ within the limits of experimental error. This would be practically equivalent to using the complete formula (b) instead of Sommerfeld's formula (e).

In order to see which formula would give us the result anticipated, namely, a constant value of s throughout the system of elements, two sets of calculations were made, one using the complete formula (b), the other using (e). The results are shown below:

Atomic No.	Element.	$\Delta\nu/R$.	s_b from (b).	s_e from (e).	$\Delta s=(s_e-s_b)\times 10^3$.
74	W	98.54	3.500	3.508	8
76	Os	111.08	3.566	3.574	8
77	Ir	118.64	3.486	3.494	8
78	Pt	125.92	3.496	3.507	11
79	Au	133.80	3.479	3.492	13
81	Tl	150.49	3.481	3.497	16
82	Pb	160.02	3.424	3.441	17
83	Bi	169.73	3.402	3.419	17
90	Th	250.86	3.390	3.428	38
92	Ur	278.73	3.435	3.486	51

Due to the smallness of the difference in (b), it was necessary to use seven-place logarithms in the calculations of s given in the fourth column. The calculations thus made gave an accuracy of about two units in the last decimal place.

The values of $\Delta\nu/R$ are taken from the precision measurements of Coster[2] for the sharpest of the L-doublets, $\beta_1 - \alpha_2$.

The variations of s_e about an average value 3.492[3] are probably due to the variations in the experimental values for $\Delta\nu/R$. It would have been better to use a weighted average from all the L-doublets for the computations. But the sixth column shows that the values of s_b are more nearly independent of Z than the values of s_e.

II. Relation of correspondence principle to expansion of $\Delta\nu/R$.—Let

[1] Duane and Patterson, Proc. Nat. Acad. Sci., 6, p. 516, 1920.

[2] Coster, Zeits. f. Phys., vol. 6, p. 190, 1921.

[3] Cf. Sommerfeld, l. c., p. 611.

us now see whether formula (*e*) is in agreement with the correspondence principle. This would mean that the fourth term of the energy difference in Eq. (*d*) must be smaller than the difference in the energy radiated per revolution in the two L-orbits, calculated according to classical theory. The energy thus calculated is

$$U = \frac{2}{3}\frac{e^2}{c^3}\int_0^\tau \dot{v}^2 dt, \tag{1}$$

where τ is the time of one revolution. For an ellipse, we may write $r = a(1 - \epsilon \cos u)$. Because of constant sectorial velocity,

$$(p/m)dt = xdy - ydx = a^2\sqrt{1-\epsilon^2}\,(1 - \epsilon\cos u)du.$$

Now since $m\dot{v} = e^2(Z - s)/r^2$,

$$\dot{v}^2 dt = \frac{e^4(Z-s)^2}{a^4(1-\epsilon\cos u)^4 m^2}\cdot a^2\sqrt{1-\epsilon^2}\,(1-\epsilon\cos u)\frac{m}{p}du,$$

and Eq. (1) becomes

$$U = \frac{2}{3}\frac{e^6(Z-s)^2\sqrt{1-\epsilon^2}}{a^2pc^3m}\int_0^{2\pi}\frac{du}{(1-\epsilon\cos u)^3}, \tag{2}$$

where u is the eccentric anomaly and p is the angular momentum, which remains constant throughout the revolution. The integral is most easily evaluated by complex methods,[1]

$$\int_0^{2\pi}\frac{du}{(a+\epsilon\cos u)^3} = \frac{1}{2}\frac{d^2}{da^2}\int_0^{2\pi}\frac{du}{(a+\epsilon\cos u)}$$
$$= \frac{1}{2}\frac{d^2}{da^2}\left[\frac{2\pi}{\sqrt{a^2-\epsilon^2}}\right] = \pi\left[-\frac{2a^2+\epsilon^2}{(a^2-\epsilon^2)^{5/2}}\right].$$

Putting $a = 1$, and $f = 1 + \epsilon^2/2 = \frac{1}{2}[3 - K^2/(n' + K)^2]$,

$$U = \frac{8\pi}{3}\alpha^3 R(Z-s)^4 h\frac{f}{K^5}. \tag{3}$$

(This is only the first term of a rapidly converging series, since m has been considered a constant in the integration.)

The energy difference between the two L-orbits is therefore[2]

$$U_1 - U_2 = (8\pi/3)\alpha^3 R(Z-s)^4 h(f_1/K_1^5 - f_2/K_2^5)$$
$$= (8\pi/3)\alpha^3 R(Z-s)^4 h(1.375 - .031), \tag{4}$$

and the ratio of the fourth term to $(U_1 - U_2)$ is

$$(h\Delta\nu)_4/(U_1 - U_2) = 3.38 \times 10^{-14}(Z-s)^6. \tag{5}$$

[1] See Sommerfeld, Atombau und Spektrallinien, Zusatz 6, p. 666.

[2] This result is given in the article by Sommerfeld and Heisenberg, loc. cit.

Hence, even for $(Z - s) = 100$, the fourth term is only a little more than 3 per cent of the energy radiated per revolution calculated according to classical theory, while the third term in the expansion is of the same order of magnitude, for the range of atomic numbers considered.

A simple calculation shows that if we neglect the fourth term for atomic numbers below 90, we should neglect the third term for atomic numbers below 50 and the second term for atomic numbers below 20. These calculations agree fairly well with the results given in the article by Sommerfeld and Heisenberg.

The author wishes to express his thanks to Professor Arnold Sommerfeld for the suggestion of the problem, and for his interest and assistance during the progress of its solution.

UNIVERSITY OF WISCONSIN,
December 18, 1922.

THE CRYSTAL STRUCTURE OF IRON-NICKEL ALLOYS.

By L. W. McKeehan.

Abstract.

Crystal structure and density of a complete series of pure iron-nickel alloys.—Fourteen alloys were prepared by melting together Armco iron and electrolytic nickel, then swaging the castings and drawing and rolling into thin tapes. X-ray diffraction patterns were obtained with these, after various heat treatments. With from zero to 25 per cent Ni, the structure is chiefly body-centered-cubic, the parameter a increasing from 2.872×10^{-8} cm for *pure iron* to about 2.89×10^{-8} cm. With from zero to 70 per cent Fe, the structure is chiefly face-centered-cubic, the parameter a increasing from 3.510 for *pure nickel* to about 3.60×10^{-8} cm. Around 30 per cent Ni, either structure is on the verge of instability and both or either may exist. Cold work tends slightly to increase the parameter. The densities of pure iron and nickel crystals come out 7.775 and 8.953 gm/cm^3, respectively.

Speculations regarding atoms of iron and nickel.—It is probable that nickel atoms differ from iron atoms in possessing six electron orbits of a certain type in place of four. The behavior of these atoms in forming crystals is consistent with the idea that the directions of closest approach in either pure metal are the normals to the planes of these orbits and that in the intermediate directions, twelve for iron and eight for nickel, the atoms cannot approach so closely, these directions determining the positions of what we may call protuberances on the atoms. The electrons which are responsible for cohesion may be not the valence electrons but those in a deeper layer.

THE alloys of iron and nickel have long possessed great interest on account of the wide range in properties covered by the series and the irreversible character of certain of the transformations. No complete discussion of the alloys will here be attempted, only the results of X-ray crystal analysis being presented at this time. The method has been fully explained elsewhere, in connection with the analysis of certain alloys of silver with gold and palladium.[1] It should be stated here, however, that the experimental work on the nickel-iron series was practically completed before that on the noble metal series was commenced. The accuracy attained was not quite as high, partly on this account and partly on account of the more frequent occurrence of relatively large grains giving discrete spots in the film.

The qualitative work of Miss Andrews,[2] who found a transition in structure in the neighborhood of Fe(75)–Ni(25), from the body-centered cubic arrangement characteristic of Fe to the face-centered cubic arrange-

[1] L. W. McKeehan, Phys. Rev., **19**, 537–538 (1922).

[2] Mary R. Andrews, Phys. Rev., **17**, 261; **18**, 245–254 (1921).

ment characteristic of Ni, is the only previous investigation in which a graded series of alloys reasonably free from impurities was used. In the present study the alloys used were made from Armco iron and electrolytic nickel melted together in a Northrup induction furnace, cast, swaged, drawn, and rolled into tape 0.015 cm thick. Chemical analyses of the castings showed that the desired compositions were nearly attained. For the X-ray analysis the tapes were further thinned by dipping in aqua regia, the best thickness being about 0.004 cm. This form of sample is well adapted to high accuracy where, as in the present case, large crystals do not occur.[1] Besides end members in this tape form, iron and nickel powders of high purity were available and were examined, packed in thin capillary glass tubes.

Three conditions were investigated for most of the samples; the first (1) was without preliminary heat treatment, the metal having been last previously passed several times through cold rolls; the second (2) was after a short anneal at 900°–950° C (except as noted) followed by slow cooling in the electric resistance furnace; and the third (3) was after a further heating to about 600° C followed by rapid cooling in air on a cold copper plate. A few of the samples were also analyzed in a fourth condition (4) obtained by chilling after (2) in liquid air for an hour or more. In Table I. are recorded the parameters a of the space lattices present, the per cent of nickel found by chemical analysis, the mean distance between the nearest atom-centers and the density ρ, computed by means of the following constants:[2]

Atomic weight of Fe	55.84
Atomic weight of Ni	58.68
Number of molecules per gram-molecule	6.0594×10^{23}
Wave-length of molybdenum $K\alpha_1$	0.70783×10^{-8} cm
$K\alpha_2$	0.71212×10^{-8} cm

The results are given graphically in Fig. 1, which shows the variation in the parameters with the atomic composition which is given in the fourth column of Table I.

The increase in the face-centered cubic space-lattice parameter of nickel due to substitution of iron is evident, although not nearly so striking as the corresponding increase in other cases since studied.[3] The increase in the body-centered cubic space-lattice of iron due to substitution of nickel is just appreciable in the short range of stability of this

[1] L. W. McKeehan, Frank. Inst. J., **193**, 231–242 (1922).

[2] L. W. McKeehan, Science **56**, 754 (Dec. 29, 1922).

[3] L. W. McKeehan, Phys. Rev., **19**, 537–538; **20**, 424 (1922). E. C. Bain, preprint No. 1139-N. Issued with Mining and Metallurgy (February, 1922); Chem. and Met. Eng. **28**, 21 (Jan. 3, 1923).

space-lattice. The parameter for iron agrees with that found by others,[1] but that for nickel is less.[2] In the latter connection it is easy to see that high values would result from most of the probable impurities. The interesting changes in crystal structure in alloys with about 30 per cent Ni were noted by Miss Andrews.[2] This region is one of great variability in space-lattice parameter, and two reasons are suggested for this fact.

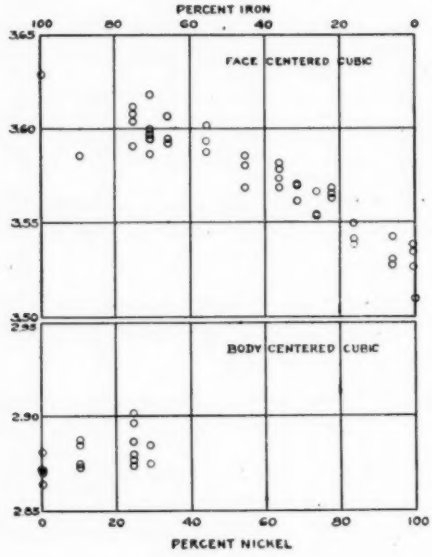

Fig. 1.

Either arrangement is on the verge of instability and correspondingly sensitive to small variations in composition or treatment, to the presence of impurities, and to similar causes. It is also possible for both phases to coexist, probably metastably, and there may be a certain degree of segregation of iron and nickel in such cases. The variability at the nickel end of the series may be attributed to impurities in the electrolytic nickel, which contained about half a per cent of other metals.

The general effect of cold-work is seen to be an increase of the parameters, due no doubt to incipient rupture in many of the crystals. The increase in crystal size due to annealing below the A_3,$(\alpha-\gamma)$ transformation is very apparent in the photographs. This transformation occurs over a very wide range in temperature in passing from pure iron to a little past 30 per cent Ni[3] and cannot be detected in these experiments at

[1] A. W. Hull, Phys. Rev., **10**, 661–696 (1917). A. Westgren, Nature, **109**, 817–818 (1922).

[2] A. W. Hull, Phys. Rev., **17**, 571–588 (1922).

[3] F. Hegg, Arch. des Sciences, **29**, 592–617; **30**, 15–45 (1910).

34 per cent Ni (atomic). Since positive errors are more probable than negative the lowest point in each vertical group is generally to be preferred.

The distances between adjacent atom-centers vary, of course, in the same way as the space-lattice parameters, but the small difference between the values obtained in the two types of space-lattice shows how nearly it is permissible to consider the atoms as spheres. While the idea is highly artificial it may be worth noting that the effect of diluting either pure metal with the other is the same as if the iron and nickel atoms were equal spheres, with, respectively, twelve and eight symmetrically distributed protuberances. In the body-centered cubic arrangement of such iron atoms, the protuberances would not lie on the lines joining adjacent atom-centers, and the parameter of the space-lattice

TABLE I.

Nominal Per Cent.		Actual Per Cent.	Atomic Per Cent.	Condition.	Body-centered cubic.			Face-centered cubic.			Note.
Fe.	Ni.	Ni.	Ni.		a.	$\frac{1}{2}a\sqrt{3}$.	ρ.	a.	$\frac{1}{2}a\sqrt{2}$.	ρ.	
100	0			2	2.872	2.488	7.775				(*a*)
100	0	0.15	0.14	1	2.881	2.495	7.704				
				2	2.872	2.487	7.779				
				2	2.864	2.480	7.848	(3.629	2.566	7.712)	(*b*)
				3	2.870	2.485	7.796				
90	10	10.68	10.20	1	2.885	2.498	7.718				
				2	2.873	2.488	7.809	(3.586	2.536	8.034)	(*b*)
				2	2.888	2.502	7.687				
				3	2.875	2.490	7.798				
75	25	25.85	24.91	1	2.897	2.509	7.676				
				2	(2.902	2.513	7.636)	3.591	2.539	8.062	(*b*)
				2	2.874	2.489	7.863	(3.608	2.552	7.944)	(*b*)
				2	2.880	2.494	7.816	(3.612	2.554	7.918)	(*b*, *c*)
				4	2.877	2.492	7.836	(3.604	2.549	7.966)	(*b*)
				4	2.887	2.501	7.752				
70	30	30.15	29.11	1				3.595	2.542	8.050	
				2				3.587	2.536	8.103	
				2				3.597	2.543	8.036	(*c*)
				4	2.875	2.490	7.868	(3.600	2.546	8.014)	(*b*)
				4	2.885	2.498	7.787	(3.618	2.558	7.898)	(*b*)
65	35	35.14	34.02	1				3.595	2.542	8.069	
				2				3.593	2.540	8.085	
				3				3.607	2.550	7.991	
55	45	45.57	44.35	1				3.602	2.547	8.063	
				2				3.588	2.537	8.163	
				3				3.594	2.542	8.117	

TABLE I (*continued*).

Nominal Per Cent.		Actual Per Cent.	Atomic Per Cent.	Condition.	Body-centered cubic.			Face-centered cubic.			Note.
Fe.	Ni.	Ni.	Ni.		a.	$\frac{1}{2}a\sqrt{3}$.	ρ.	a.	$\frac{1}{2}a\sqrt{2}$.	ρ.	
45	55	55.82	54.61	1				3.586	2.535	8.217	
				2				3.581	2.532	8.245	
				3				3.569	2.524	8.330	
35	65	64.82	63.70	1				3.569	2.524	8.371	
				1				3.574	2.527	8.335	
				2				3.582	2.533	8.279	
				3				3.579	2.531	8.301	
30	70	69.60	68.54	1				3.571	2.525	8.374	
				2				3.570	2.525	8.381	
				3				3.562	2.519	8.441	
25	75	74.51	73.76	1				3.567	2.522	8.424	
				2				3.555	2.514	8.510	
				3				3.554	2.513	8.508	
21	79	78.54	77.69	1				3.569	2.524	8.429	
				2				3.566	2.522	8.446	
				3				3.564	2.520	8.465	
15	85	84.48	83.83	1				3.550	2.511	8.586	
				2				3.542	2.505	8.647	
				3				3.539	2.502	8.672	
5	95	94.24	93.96	1				3.531	2.497	8.774	
				2				3.543	2.505	8.682	
				3				3.528	2.495	8.795	
0	100	99.50	99.47	1				3.539	2.503	8.736	
				2				3.527	2.494	8.824	
				3				3.535	2.500	8.765	
0	100			2				3.510	2.482	8.953	(*a*)

Notes: (*a*) Pure powdered metals, glowed in reducing atmosphere.
(*b*) Values in parentheses due to few and faint lines.
(*c*) Annealing temperature 700° C in place of 900°–950° C.

would be the same as if the protuberances did not exist, but nickel atoms substituted for any of the iron atoms must present protuberances to some or all of its neighbors and thus distend the space-lattice. The same argument would explain the effect of iron atoms upon the space-lattice of nickel, and the effect would here be more positive because the packing is already closer, with less room for interatomic adjustment. The atomic number of iron is 26, that of nickel 28, so that there are two more electrons in the latter than in the former. If the crystal is held together by forces acting along the lines joining adjacent atoms, it is perhaps per-

missible to regard the eight neighbors of each iron atom as attracted by the eight faces of four elliptical electronic orbits, and the twelve neighbors of each nickel atom by the twelve faces of six elliptical electronic orbits. The twelve symmetrically arranged directions farthest from the eight bonds in the case of iron, the eight symmetrically arranged directions farthest from the twelve bonds in the case of nickel would correspond to the protuberances in the more mechanical picture first suggested.

It is suggested that the electrons responsible for the cohesion of metallic crystals are not the so-called valence electrons but those of a deeper layer relatively more firmly attached to the nucleus, the external electrons being left free in the sense required for electrical conductivity. We are already accustomed to the idea that nominally external electrons may penetrate into nominally internal layers so this idea is not at all revolutionary. The comparative weakness of metal crystals would be a natural consequence of this less intimate connection between adjacent atoms which would thus form a transition stage between valence-held salt-like crystals and amorphous liquids.

RESEARCH LABORATORIES OF THE
AMERICAN TELEPHONE AND TELEGRAPH COMPANY, AND THE
WESTERN ELECTRIC COMPANY, INCORPORATED,
November 21, 1922.

A METHOD FOR THE NEUTRALIZATION OF ELECTRON SPACE CHARGE BY POSITIVE IONIZATION AT VERY LOW GAS PRESSURES.

By K. H. Kingdon.

Abstract.

Neutralization of space charge around a hot filament by imprisoned positive ions in gas at very low pressures.—(1) *Design of tube.* If a very small filament, diameter 0.01 cm, is run axially through a cylindrical anode with closed ends, positive ions formed between the electrodes can only rarely escape and will describe orbits around the filament until they lose sufficient energy by collision with gas molecules to enable them to fall into the cathode. The imprisoned ions, during their lives, neutralize a certain amount of the space charge between the electrodes. The effect should increase with the absolute temperature and with the ratio of the cross-section of the anode cylinder to that of the filament. A theoretical calculation indicates that in He at 10^{-6} mm, an ion which misses the filament on its first passage across the tube may circulate around the filament 300 times before discharging to the cathode. (2) *Results in He, H, Ne, and Hg* at pressures ranging from 10^{-2} to 10^{-7} mm are shown in curves. Comparison of the currents when the cylinder ends were connected (1) to the anode and (2) to the cathode gave the effect of the positive ions, and also their number, and therefore the increase in current per ion, α. At pressures so low that the 3/2 power space-charge law holds in an ordinary tube, imprisoned ions may still produce large deviations from this law, in favorable cases the current with the ions being 5 or 10 times the current as ordinarily limited by space charge. The relatively much greater effect in Hg vapor than in He is shown to agree well with the theory. If α depended only on mean free path, however, it should vary as $1/p$, but the results gave $(1/p)^{2/3}$. This difference shows the influence of other factors. However, theoretical mean free paths are of the same order of magnitude as those calculated from α. Moreover, in Hg vapor at 4.2×10^{-7}mm, the time required to reach equilibrium after the positive ions began to accumulate was found by oscillograms to be about the same as the calculated life of an ion, 1.4×10^{-3} sec. The simple theory, then, seems to be pretty well verified.

I. Introduction.

THE residual gas pressure in a high vacuum thermionic device is usually less than 10^{-4} mm Hg, and the normal free paths of the electrons are of the order of several hundred centimeters. The few positive ions formed carry an insignificant fraction of the current across the tube and cannot ordinarily contribute much to the neutralization of electron space charge as they soon discharge to the cathode or move out from the space between the electrodes.

If, however, the cathode is a straight wire, and the anode is a coaxial cylinder of very much larger radius, many of the positive ions formed near the anode are unable to reach the cathode on the first passage across the tube, because of their initial velocity transverse to the radius due to thermal energy. The escape of such ions from between the electrodes can be prevented to a great extent by making the anode a cylinder with closed ends, through which small holes are pierced to admit the cathode filament. An ion imprisoned in this way executes orbits to and fro across the tube, until in some way it loses its velocity transverse to the radius, and then discharges to the cathode.

II. Theory.

The general case of the motion of an ion in electric and magnetic fields between coaxial cylinders has been investigated by Hull.[1] We shall take the case in which there is no magnetic field and the radius of the anode cylinder is much greater than that of the cathode. Following Hull's notation let V be the potential difference between the electrodes; R and r be the radius of anode and cathode, respectively; e and m be the charge and mass of an ion; and v_0 be the initial velocity of the ion transverse to the radius. The initial radial velocity of the ion can be neglected as it is small compared with the velocity acquired in falling through the potential difference V.

The condition[2] that a positive ion starting from a point immediately inside the anode shall just fail to strike the cathode is that V shall be less than the value given by $eV = 1/2mv_0^2(R/r)^2$.

From this equation we see that:

(1) For given V and R/r the same proportion of ions formed in any gas will fail to reach the cathode, since $1/2mv_0^2$ is the same for all gases at the same temperature.

(2) The number of ions missing the filament increases rapidly with increase of R/r over a certain range of values of this ratio.

(3) The higher the temperature of the gas in the tube, the greater will be the value of $1/2mv_0^2$, and the greater will be the proportion of ions missing the cathode.

(4) The value of R/r necessary so that the average ion will just miss the cathode increases as the square root of V.

III. Experiments.

Apparatus.—Most of the experimental work was done with two tubes denoted respectively by K-26 and K-27.

[1] A. W. Hull, Phys. Rev., 18, 31 (1921).

[2] From Eq. (10) of Hull's paper, putting $H = 0$.

Tube K-26 had a cathode C (Fig. 1) of 4 mil (0.01 cm) tungsten wire, and an anode A of thin molybdenum, 0.75 inch (1.9 cm) in diameter, and 1.4 inches (3.6 cm) long. Two guards to imprison the ions between A and C were mounted at GG. In this tube the guards were made of six spokes of 10 mil tungsten wire evenly spaced around a nickel ring, 1.9 cm in diameter. The spokes did not reach the centre of this ring, and left a space 2 mm in diameter for the cathode to pass through. The filament was welded to stouter leads outside these guards. The electrode system was assembled on a glass frame and then mounted as one unit inside a hard glass tube.

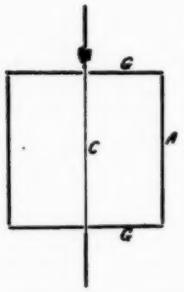

Fig. 1. Design of experimental tube.

Tube K-27 was of the same general construction. It had a 4 mil filament and an anode cylinder 2 inches (5.1 cm) in diameter and 2 inches long. The guards were in this case molybdenum discs, 2 inches in diameter, pierced with 52 mil (0.13 cm) holes to admit the cathode filament.

Gas manipulation.—The tube was thoroughly baked out, and the electrodes bombarded. The desired gas was admitted, and measurements made with the tube shut off by a mercury seal between the liquid air trap and the pump. The higher gas pressures were measured with a McLeod gauge, and the lower with an ionization gauge.

The gases used were helium, hydrogen, neon, and mercury vapor. The first three were kept in contact with charcoal cooled in liquid air throughout the experiment. The pressure of mercury vapor was determined from the temperature of a U tube in which mercury vapor was condensing continually from a bulb at higher temperature. The pump was kept running during the measurements with mercury.

Typical measurements.—Two types of volt-ampere characteristics, as a rule, were taken at each pressure. In the first the guards were connected to the negative end of the filament ($V_g = 0$), and any positive ions formed could therefore discharge at the ends of the cylinder. In the second type of characteristic the guard plates were connected to the anode cylinder, and the escape of positive ions through the ends of the cylinder was thus prevented.

Examples of both types of characteristic are given in Figs. 2 and 3.

Fig. 2 shows the effect of neon in tube K-26. The lower curves are with the guard plates at zero volts ($V_g = 0$), and in the upper curves they are connected to the anode cylinder ($V_g = V_p$). At a pressure of $13(10)^{-5}$ mm the tube shows a fairly good vacuum characteristic with $V_g = 0$, the logarithmic plot of current against voltage between the

anode and the negative end of the filament, being approximately a straight line. With $V_g = V_p$ the characteristic shows the effects of

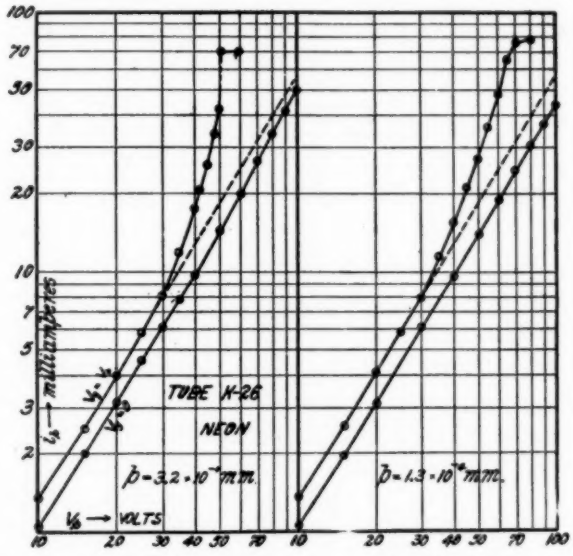

Fig. 2. Logarithmic plot of volt-ampere characteristics of tube K-26 at low pressures of neon.

positive ionization markedly above 30 volts. The $V_g = 0$ characteristic lies below the other on account of the electrostatic screening action of the guards when they are connected to the filament.

Fig. 3 shows the effect at various low pressures of mercury vapor in tube K-27. With this tube the effect is much stronger than with K-26 on account of the greater ratio of anode to cathode diameters. The pressures of mercury vapor ranged from 10^{-7} mm to 10^{-6} mm. The lowest curves, as before, were taken with $V_g = 0$, and hardly show the effects of positive ionization. They lie below the others on account of the electrostatic screening action of the guards, which is much greater in this case than before on account of the greater diameter of the cylinder.

These figures illustrate the point that a gas pressure which in an ordinary electron discharge tube is insufficient to disturb the

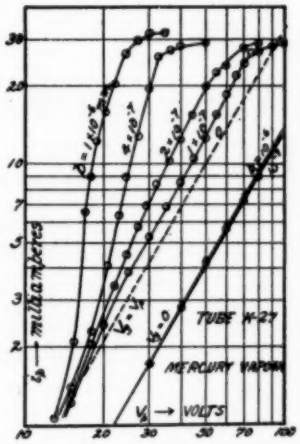

Fig. 3. Logarithmic plot of volt-ampere characteristics of tube K-27 at low pressures of mercury vapor.

limitation of current by electron space charge may produce almost complete elimination of space charge if the tube is designed so as to imprison the positive ions between the electrodes.

Some measurements were made to estimate the amount of positive ionization occurring in each tube at various gas pressures. The guard plates were made negative to the cathode, and the positive ion current flowing to them measured with various voltages applied to the cylinder. The negative voltage was made high enough to obtain an approximately saturated positive ion current. It should be noted in this connection, however, that practically only those positive ions were collected which missed the cathode on their first passage across the tube, and subsequently drifted towards the ends; for the field due to the guard plates would not be strong enough to pull the other ions out from the center of the cylinder before they struck the cathode. These measurements of ionization current are however significant for our purpose, for it is precisely those ions which do not discharge to the cathode on their first passage across the tube which are most effective in neutralizing electron space charge.

IV. Discussion of Results.

In Figs. 2 and 3 the dotted lines give the current which would flow across the tube in a perfect vacuum with $V_g = V_p$. At any voltage, the difference between the ordinates of the full curve and the dotted line represents the extra electron current which flows as the result of the neutralization of space charge by positive ionization. Denote this increase by Δi_p; and let the positive ion current associated with it be i_g. Then the space-charge effect of Δi_p electrons is neutralized by i_g positive ions, and hence the ratio $\Delta i_p / i_g = \alpha$ represents what may be termed the effectiveness of a positive ion in neutralizing electron space charge.

The factors which limit the lives of the positive ions between the electrodes, and consequently limit the value of α are as follows:

(1) Discharge of the ions to the cathode;
(2) Leakage of the ions through the holes in the guard plates;
(3) Recombination with electrons;
(4) Effects of the collisions of ions with gas molecules.

Rough calculations show that the effects of (2) and (3) are small. Some of the ions strike the cathode on their first passage across the tube. Those that do not should continue to oscillate back and forth across the tube until their orbits are disturbed in some way.

The experimental evidence seems to point to (4), the collisions between ions and molecules, as being the chief cause limiting the lives of the positive ions. The value of α is found to be very closely connected with

the gas pressure, and consequently with the mean free path of the ions. This is exemplified in Fig. 4 which shows the logarithm of α plotted against the logarithm of the reciprocal of the gas pressure in microns Hg. The data shown were obtained with helium in tube K-27, with 35 volts

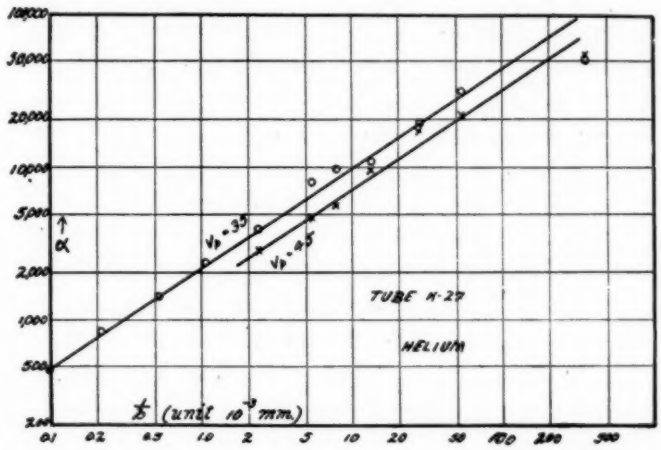

Fig. 4. Logarithmic plot showing the relation between the reciprocal of the pressure and the number of electrons whose space-charge effect is neutralized by one positive ion.

on the cylinder for the upper curve and 45 volts for the lower. The points lie approximately on a straight line of slope 2/3, and the existence of this systematic relation between α and $1/p$ indicates that there is a close connection between the lives of the ions and their mean free path through the gas. The free path cannot, however, be the only factor entering, for, if it were, the slope of the curves in Fig. 4 should be 1 instead of 2/3. There seems to be no simple theoretical reason why the 2/3 power of $(1/p)$ should replace the first, and this difference is probably due to the other factors limiting the lives of the ions as outlined above.

Referring to the previous enumeration we may say that as regards factor (3) a constant fraction of the ions formed at any pressure will recombine with electrons. This cause therefore cannot account for the change in slope. The second factor, however, provides a correction of the right sort, for the rate of flow of ions through the guards will be nearly independent of the gas pressure, and the ions lost in this way will be a larger fraction of the total number formed at low pressures than at high. A rough estimate of this leakage of ions gave a correction of the right kind and one which became quite important at the lower pressures used. Again, as regards factor (1), a constant fraction of the ions formed at any pressure will discharge to the cathode on their first passage across the tube. The ions measured in i_g were only those which missed

the cathode on their first transit, whereas both classes of ions contribute to the neutralization of space charge. It is evident that the contribution of the ions striking the cathode becomes more important the higher the pressure for at these pressures no ion can live long in the tube. This means that the values of α obtained at the higher pressures were too large, and the leakage correction shows that the values of α found at low pressures are too small. If these corrections could be allowed for, the slope of the curve in Fig. 4 would be increased, probably from 2/3 to 1.

Comparison with kinetic theory.—From the electrical experimental data it is possible to make an estimate of the distance traversed by a positive ion during its life between the electrodes, and this may be compared with the average free path of an ion calculated from kinetic theory.

Take the case of a helium ion in tube K-27 starting from a point just inside the anode and moving inwards under a potential difference of 35 volts between anode and cathode. Approximate calculation shows that the time (τ_+) taken to move from the anode to the point of the path closest to the cathode is about 1.4×10^{-6} sec. A similar calculation for the time (τ_-) taken for an electron to move from cathode to anode under a potential difference of 35 volts gives 1.08×10^{-8} sec. Now τ_+ and τ_- give the order of magnitude of the contributions to space charge of an ion and an electron for one passage each between cathode and anode. Hence one ion moving once from anode to cathode will neutralize the space charge of $\tau_+/\tau_- = 130$ electrons, and will permit this number of electrons in excess of the normal electron space-charge current to flow across the tube. It is worth noting that τ_+/τ_- is somewhat greater than the square-root of the ratio of the masses of a helium ion and an electron (86) which would determine their relative velocities after falling through the same potential difference. This is due to the potential distribution between the electrodes.

We have seen above that actually α electrons are released per positive ion. Hence $\alpha/(\tau_+/\tau_-)$ must measure the number of times which a positive ion crosses the tube; and if R be the radius of the anode, $\alpha R(\tau_-/\tau_+) = L_1$ is the distance travelled by the ion before discharging to the cathode.

The average distance (L_2), calculated from kinetic theory, which an ion travels before colliding with a gas atom may be found by multiplying the average free path of a gas molecule by $\sqrt{2}$ to take account of the high velocity of the ion. The temperature of the gas was estimated at 500° K. Fig. 5 shows the logarithms of L_1 and L_2 plotted against the logarithm of the reciprocal of the pressure. The free paths calculated in the two ways are seen to be of the same order of magnitude. The smaller slope of the L_1 curve is probably accounted for by the factors discussed for Fig. 4.

The calculation of L_1 is not sufficiently precise to determine exactly how many collisions with gas atoms an ion makes on the average before discharging to the filament. In order to strike the filament the component of velocity of the ion transverse to the radius must be small,

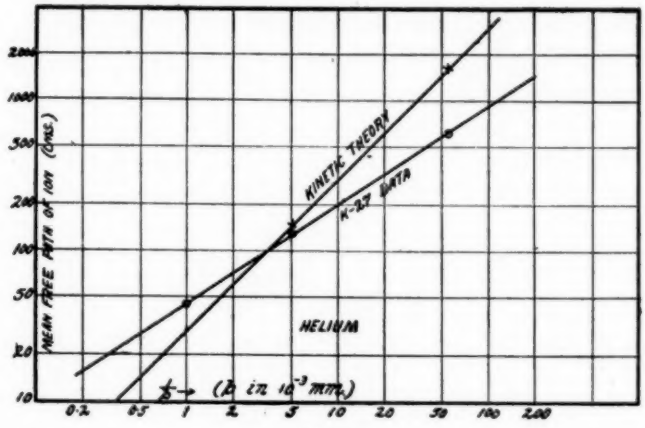

Fig. 5.

and the effect of collisions must be to reduce this component to the necessary value. Horton and Davies,[1] and Saxton[2] have found that positive ions lose energy on colliding with gas molecules. In the present experiments it seems likely that at the first collision the ion will lose a considerable part of its energy, and will not be able to move out as close to the anode as it did before. Its total velocity will be smaller, but its transverse velocity probably will be larger than before the collision. It is therefore unlikely that the ion will strike the filament immediately after the first collision. Succeeding collisions reduce the energy of the ion still further, and force it to follow paths which on the average get closer and closer to the filament. Finally after perhaps three or four collisions the total energy of the ion is so low that it is able to strike the cathode.

Comparison of results for mercury and helium.—The first experiments made with mercury vapor showed at once that a low pressure of mercury gave as great neutralization of space charge as a much higher pressure of helium. This difference is due partly to the lower ionization potential of mercury and partly to the great mass and low velocity of the mercury ions. More ions are formed in mercury at a given pressure than in helium, and each ion stays in the tube for a longer time. From the experimental data we can calculate how long each kind of ion stays in

[1] Horton and Davies, Proc. Roy. Soc., **95**, 349 (1919).

[2] Saxton, Phil. Mag., **44**, 822 (1922).

the tube, and can see what relation this bears to its effectiveness in neutralizing electron space charge. A comparison of the two gases was carried out in this way, for the difference between their effects was so striking that it was thought at first that some new factor must enter in the case of mercury. The comparison showed that the simple theory of the effect described above accounted quite satisfactorily for the difference in the results obtained with the two gases.

The mean free path of an ion at a given pressure was calculated by the kinetic theory formula, and this distance was divided by the average velocity of the ion under the electric field. This gave a time proportional to the life of the ion and this time should be proportional to the effectiveness of the ion in neutralizing space charge as measured by α.

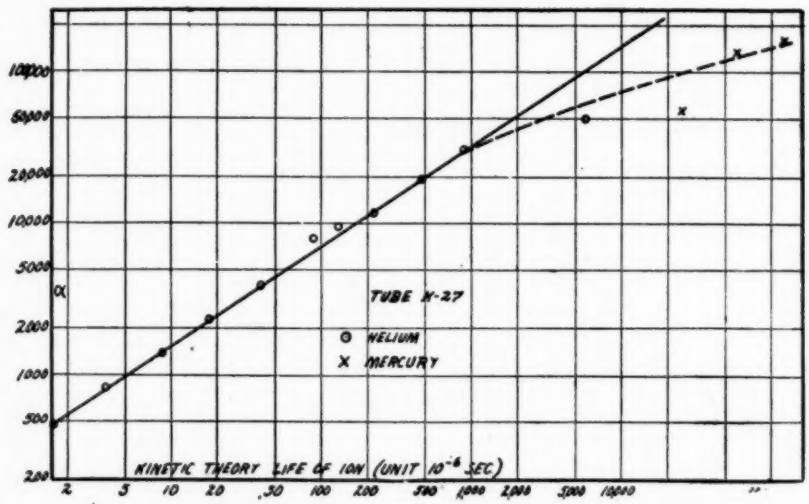

Fig. 6. Comparison of effectiveness of mercury and helium ions in neutralizing electron space charge.

In Fig. 6 the logarithm of the interval between collisions is plotted against the logarithm of α. The points obtained from the helium data lie on a line of slope 2/3 as before (Fig. 4). The mercury points do not fall on this line but lie considerably below it. The agreement is, however, sufficiently good to show that the effectiveness of any kind of ion in neutralizing space charge is proportional to its mean free path and the reciprocal of its velocity. The probable reason why the mercury points lie below the line is as follows. The pressures of mercury used were very low, ranging from 10^{-6} to 10^{-7} mm. Observations could not be made at higher pressures as the plate current reached saturation. The pressures were calculated from the temperature of the mercury, and this

made no allowance for the presence of residual gas, the pressure of which was probably of the order 10^{-6} mm. The mercury ions were therefore not able to traverse their calculated free paths. The escape of the ions through the end guards would also tend to lower the values of α at such low pressures.

The paths of the ions.—A rough calculation of the path of an ion moving towards the cathode was made from the equations of motion of the ion. Fig. 7 shows the path of a positive mercury ion moving from just inside an anode cylinder of 1 cm radius towards an axial cathode, under a potential difference of 35 volts. The initial thermal velocity of the ion was taken as that due to a temperature of 300° K and was assumed to be all transverse to the radius.

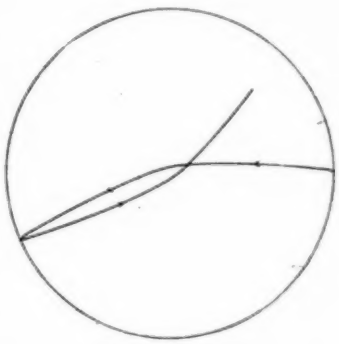

Fig. 7.

From the experimental data for tube K-27 we can form an estimate of the number of times which an ion crosses the tube. Let us assume that all the ions come from near the anode, and that by "crossing the tube" we mean that an ion moves from near the anode to near the cathode. Each electron crosses the tube only once. Then as shown above, $\alpha r_- / r_+$ will give the number of times which an ion crosses the tube. For helium in tube K-27 this becomes $7.7(10)^{-3}\alpha$ for an anode voltage of 35.

Fig. 8 shows the logarithm of the number of times of passage plotted against the logarithm of the reciprocal of the pressure for helium. Thus

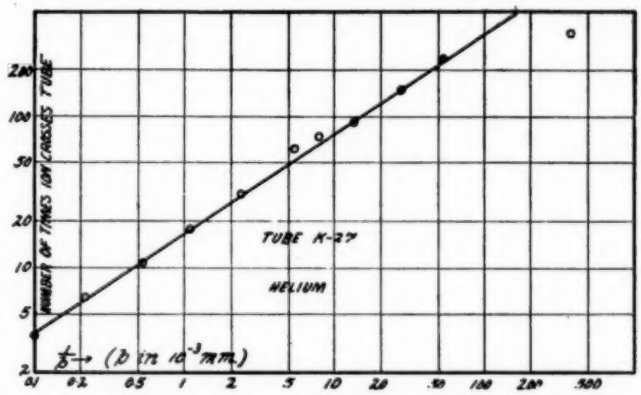

Fig. 8. Logarithmic plot showing number of times which an imprisoned helium ion crosses tube K-27 as a function of gas pressure.

at a pressure of 10^{-5} mm a helium ion may cross the tube 350 times before losing its energy, and discharging to the cathode.

Check by oscillograph records.—If the guard plates in tube K-27 are initially at the same potential as the cathode and then are abruptly raised to the anode potential, the number of positive ions imprisoned between the electrodes will increase to an equilibrium value at which the number of ions formed per second is equal to the number disappearing. The time taken for reaching this equilibrium will be of the same order of magnitude as the life of a single ion. Therefore if the anode voltage is applied intermittently to the guards by a commutator, and an oscillogram is taken of the corresponding changes in anode current, the time required for this current to reach say half its maximum value will be of the same order of magnitude as the life of an ion in the tube.

Oscillograms were taken at low pressures of mercury vapor, and they showed the time-lag of the anode current in rising to its maximum value very clearly. The initial sudden rise in the current due to the electrostatic action of the guards when the anode voltage was applied to them could be distinguished easily on the records, from the slower rise due to the neutralization of electron space charge by the gradual accumulation of positive ions.

The lives of the ions measured in this way were of the same order of magnitude as those calculated by the other methods given above. Thus with mercury vapor at a pressure of 4.2×10^{-7} mm and an anode voltage of 30, the time taken for the anode current to reach half-value was 7.8×10^{-3} sec. whereas the calculated life of an ion under the same conditions was 1.4×10^{-3} sec.

It is a pleasure to express my thanks to Dr. Irving Langmuir for many valuable suggestions during the course of the work. The theory developed here is based on his suggestion that the effects observed were due to imprisoned positive ions.

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October 18, 1922.

THE EFFECT OF SPACE CHARGE AND INITIAL VELOCITIES ON THE POTENTIAL DISTRIBUTION AND THERMIONIC CURRENT BETWEEN PARALLEL PLANE ELECTRODES.

By Irving Langmuir.

Abstract.

Effect of space charge and cathode temperature on thermionic current and potential distribution.—I. *Case of parallel plane electrodes.* (*a*) *Current limited by space charge.* The results obtained by E. Q. Adams (unpublished), Epstein, Fry and Laue are discussed and summarized, certain errors are pointed out, and the equations are put in a form adapted to easy numerical calculation. Assuming the normal components of the velocities of the emitted electrons have the Maxwell distribution, the integration of Poisson's equation between proper limits leads to a numerical relation between the new variables $\xi = 2(x - x_m)[2\pi^3 e^2 i^2 m/k^3 T^3]^{1/4}$ and $\eta = e(V - V_m)/kT$, where x_m and V_m give the position and voltage of the plane of minimum potential, and k is the Boltzmann gas constant. Denoting values at the cathode by the subscript 1, and inserting values of constants: $\eta_1 = \log (i_0/i)$, where i_0 is the saturation current; $V - V_1 = T(\eta - \eta_1)/11{,}600$; $\xi - \xi_1 = 9.180 \times 10^5 T^{-3/4} i^{1/2}(x - x_1)$. These equations and *the tables of* $\xi(\eta)$ for various values of η enable, for a given cathode temperature T, the potential distribution for a given current i, or vice versa, to be computed. An approximate solution for the current is: $i = [(2^{1/2}/9\pi)(e/m)^{1/2}(V - V_m)^{3/2}/(x - x_m)^2](1 + 2.66\eta^{-1/2})$, which reduces to the usual three halves power law equation if we neglect V_m and x_m and the correction factor in η. (*b*) *Equilibrium condition with anode at great distance, current zero.* If the only retarding field is that of the space charge, the density of charge is: $\rho = kT\rho_1/[(kT)^{1/2} + x(2\pi e\rho_1)^{1/2}]^2$ where $\rho_1 = i_0[2\pi m/kT]^{1/2}$. Except near the cathode this is approximately equal to $kT/2\pi ex^2$; hence ρ is proportional to the absolute temperature of the cathode and inversely proportional to the square of the distance away. The potential gradient at the cathode is $X_1 = [8\pi\rho_1 kT/e]^{1/2}$. Equations are also given for the case where an external retarding field X_∞ is applied. II. *In the case of concentric cylindrical electrodes,* the current is: $i = (8^{1/2}/9)(e/m)^{1/2}[V - V_m + \frac{1}{4}V_0\{\log (V/\lambda V_0)\}^2]^{3/2}/r$, where V_0 is the initial energy of the electrons expressed in volts ($3kT/2e$), r is the radius of the anode, and λ is a constant between 1 and 2, not yet experimentally determined. The deviations from the three halves power law are not more than one quarter as much as for parallel planes and amount to only about 3 per cent at 130 volts.

THE effect of space charge on the potential distribution near an electron-emitting plane cathode was calculated by Richardson [1] for the case where the opposing electrode is at infinite distance and the

[1] Phil. Trans., A, *201*, 516 (1903).

potential gradient at this second electrode is zero. The electrons were assumed to be emitted with velocities distributed in accordance with Maxwell's Law. Under these conditions no current flows between the electrodes. Child [1] and Langmuir [2] treated the case where current flows under the influence of an accelerating field insufficient to cause saturation. Neglecting the effect of the initial velocities of the electrons they found that the current varied with the three halves power of the potential difference between anode and cathode and that the potential varied with the four thirds power of the distance from the cathode.[3]

Schottky [4] made approximate calculations for the case of small currents, taking into account the effects of initial velocities. Laue [5] gives an exhaustive treatment of the case where one or more heated electrodes (parallel planes, concentric cylinders, spheres, etc.) are in equilibrium with an electron atmosphere, *i.e.*, when no current flows.

Epstein [6] gives a clear and complete treatment of the effect of initial velocities for parallel plane electrodes when the current is limited by space charge and tabulates a function by which these calculations are facilitated. Unfortunately Epstein has committed either an error or an oversight in the use of the Boltzmann constant k so that it becomes necessary to substitute $2k$ in place of each k that occurs in his equations in order to be able to use correctly the customary value $k = 1.37 \times 10^{-16}$ erg per degree. Since Epstein gives no numerical values of the constants

[1] PHYS. REV., *32*, 492 (1911).

[2] PHYS. REV., *2*, 450 (1913); Phys. Zeitschr., *15*, 348 (1914).

[3] Lilienfeld (PHYS. REV., *3*, 364, 1914; and many subsequent papers) claims to have discovered the three halves power law in some of his work in 1910. A very careful study of Lilienfeld's 1910 paper (Ann. d. Physik, *32*, 674, 1910), made in connection with an Interference before the U. S. Patent Office (Interference No. 40380, Arnold *vs.* Langmuir, Langmuir Record, pages 352 to 395) shows that in Lilienfeld's experiments the current did not even approximately vary with the three halves power of the voltage. The original data upon which Lilienfeld bases his claim are those given on page 698 of his 1910 paper. It there appears that no current flowed until the difference of potential between the sounding electrodes was 102 volts. When this voltage was raised to 116 volts the current increased 17-fold or with the 22d power of the voltage instead of the three halves power.

Dr. A. W. Hull and the writer have constructed and studied a tube as nearly as possible identical with Lilienfeld's tube of 1910, with the result that we have found that the type of discharge observed by Lilienfeld depends upon secondary electron emission from the walls of the narrow glass tubes under the influence of electron bombardment. Undoubtedly traces of residual gas are essential in starting the discharge, but it has not proved possible to stop the discharge by improvement of the vacuum. These experimental results are described in the Langmuir Record, p. 382. A discussion and summary have been published by the writer elsewhere (General Electric Rev., *23*, 513, 1920).

[4] Phys. Zeitschr., *15*, 526 (1914); Ann. der Physik, *44*, 1011 (1914).

[5] Jahrb. d. Radioakt. u. Elektronik, *15*, 205 (1918).

[6] Ber. d. Deut. phys. Ges., *21*, 85 (1919).

in his equations, his unusual value for k must lead to error in all numerical calculations unless the whole laborious derivation is gone through.

Fry [1] treats exactly the same problem as Epstein apparently without knowledge of this earlier work and obtains essentially similar results. Fry's equations, however, are expressed in very unusual nomenclature and this makes it not only difficult for others to apply his results but has caused Fry himself to commit serious errors in all his numerical calculations. Thus $\bar{v}_0$ is defined (p. 444) as the *average* velocity component of the *emitted* electrons normal to the surface. Nowhere does he state how this velocity may be calculated from the temperature. As a matter of fact this kind of an average is totally different from those customarily used in the kinetic theory. Before coming to his final equations Fry replaces $\bar{v}_0$ by another quantity which is defined as the "potential change, $\overline{V}_0$, which would give to an electron an energy equal to the average energy of those shot out from the cathode." Now the "*average energy*" of the electrons is not equal to the energy of an electron moving with the average velocity $\bar{v}_0$ nor is it four times this energy as is implied by the second of Fry's equations on p. 449. The third equation is also in error probably due to confusion between the many possible kinds of averages. These errors are all easily avoided if the temperature is brought into the equations at an early stage.

In the summer of 1913 at the request of the writer, Dr. E. Q. Adams undertook an analysis of the space charge problem, taking into account the initial velocities in accordance with Maxwell's Law. He arrived at the complete and correct solution but unfortunately failed to publish his results. In view of the discrepancies which have occurred in the publications of Epstein and Fry, it seems desirable to summarize and compare the results of these three sets of calculations and to present the equations in a form adapted to numerical computation. At the same time it is worth while to correlate these with the equations developed by Richardson and Laue. No attempt will be made to give complete derivations of the equations for these are very satisfactorily given in the publications referred to.

Case I. Current Limited by Space Charge.[2]

We will consider here the case treated by Epstein, Fry and Adams. Electrons are emitted in accordance with Maxwell's Law from a plane cathode of infinite extent. The anode is a parallel plane surface at a positive potential (with respect to the cathode) such that the current is less than the saturation current.

[1] Phys. Rev., *17*, 441 (1921).

[2] The nomenclature adopted is essentially that of Epstein except where confusion might arise because of a different meaning given to the same symbol by Fry.

In accordance with the usual derivation of the Richardson equation, let us assume that each unit of volume of the metal contains N_v electrons. Then Maxwell's Law states that the number dN_v of electrons in this volume having velocity components (in the direction of the x-axis) lying between v and $v + dv$ is

$$dN_v = N_v \sqrt{\frac{m}{2\pi kT}} \epsilon^{-\frac{mv^2}{2kT}} dv. \tag{1}$$

Here m is the mass of the electron (9.01×10^{-28} g.); k is the Boltzmann constant (1.372×10^{-16} erg per degree), and T is the absolute temperature of the cathode.

The form of assumption above, which is that made by Epstein, is open to the objection that it postulates knowledge that we do not possess of the internal structure of the metal of the cathode. The assumption however is mathematically equivalent to the following which is free from this objection and which has received a certain amount of experimental verification on the part of Richardson and others. Let N_s be the number of electrons emitted per unit time per unit area from a plane surface. Then according to Maxwell's Law [1] the number dN_s of electrons emitted per unit time per unit area which have velocity components normal to the surface lying between v and $v + dv$ is

$$dN_s = N_s \frac{mv}{kT} \epsilon^{-\frac{mv^2}{2kT}} dv. \tag{2}$$

If the current flowing to the anode is less than the saturation current (corresponding to N_s) it is evident that this must be due to a retarding potential gradient close to the surface of the cathode by which the more slowly moving electrons are forced back to the cathode. If the potential of the anode is positive, there must then be a surface between the cathode and anode at which the potential is a minimum. Using x as abscissa to measure distances in a direction normal to the cathode surface, we let x_1, x_2 and x_m be the abscissas of the cathode, anode and the surface of minimum potential respectively. Similarly, if V is the potential at any surface represented by the abscissa x, then V_1, V_2 and V_m are respectively the potentials of the cathode, anode and surface of minimum potential.

Let i_0 be the saturation current from the cathode obtainable by use of higher anode potentials. Electrons corresponding in number to i_0 are being *emitted* continuously from the cathode even when the current is not saturated, but a certain fraction of them are then made to return by the retarding field. The actual current i which flows between cathode and anode consists of those electrons which are emitted with sufficient

[1] See Richardson, Emission of Electricity from Hot Bodies, 1916, page 141.

velocity components to enable them to move against the potential difference $V_1 - V_m$.

By means of Equations (1) or (2) the following relation is obtained between i, i_0 and $V_1 - V_m$:

$$i = i_0 \epsilon^{-\frac{e(V_1 - V_m)}{kT}}, \tag{3}$$

where e is the negative charge on the electron (*i.e.*, 4.774×10^{-10} e.s.u.).

The treatment of the problem by Adams, Epstein and Fry is based on the following facts. Between x_1 and x_m there are two groups of electrons, those moving away from and those moving back towards the cathode. Among the former all velocities from zero to ∞ are present in accordance with Maxwell's Law. Although each electron loses velocity as it moves through the retarding field, the average velocity of all the electrons at any place remains constant because the more slowly moving electrons are continually being sorted out and sent back to the cathode. Considering only velocity components normal to the surface, among the electrons returning to the cathode, we see that all velocities are not present, for the velocity acquired by the electrons at any place of potential V cannot exceed that corresponding to a fall through the potential difference $V - V_m$. Between x_m and x_2 electrons are moving only away from the cathode. At any point of potential V the normal velocities may have any value above a certain minimum corresponding to a potential difference $V - V_m$.

The above conditions regarding the distribution of velocities are taken into account by means of proper choice of the limits in the integration of Eqs. (1) or (2).

The first step in the mathematical treatment is to calculate the space charge or the electron density at any point by means of the integrations just referred to. Then by Poisson's equation

$$d^2V/dx^2 = -4\pi\rho. \tag{4}$$

To determine the potential distribution, two more integrations must be carried out. The first integral can be expressed in terms of the Probability Integral, but the second requires the numerical calculation of a new function. In connection with these integrations it is necessary to choose values for the integration constants. For the first integration the conditions imposed are:

I. $\quad dV/dx = 0 \quad$ when $\quad V = V_m$;

and for the second integration:

II. $\quad V = V_m \quad$ when $\quad x = x_m$.

The first of these conditions makes it impossible to apply the resulting equations to cases in which there is no potential minimum between the cathode and anode. For example the potential distribution between two electrodes with a potential slightly *greater* than that needed to give saturation cannot be calculated by this method, for the condition I. is not fulfilled. If in this case we impose the proper conditions for fixing the integration constants even the first integration cannot be performed. Similarly the equations resulting from these calculations must not be used where the retarding potential gradient extends up to the surface of the anode.

Before carrying out the numerical calculations required in the second integration it is desirable to reduce the equations to a form in which only pure numbers occur. This may be accomplished by introducing the new variables:

$$\eta = e(V - V_m)/kT, \tag{5}$$

$$\xi = 4(\pi/2kT)^{3/4}m^{1/4}(ei)^{1/2}(x - x_m). \tag{6}$$

It is seen that η is a measure of the potential of any point with respect to that of the minimum potential surface, this potential difference being measured in a new kind of unit. Similarly ξ measures the distance of any point from the minimum potential surface in units whose magnitude depends upon the current and the temperature.

The quantities η and ξ are the same as those represented by the same symbols by Fry although he expresses them in equations quite different from (5) and (6). Adams uses exactly the same variables in his calculations. Epstein puts his equations in slightly different form, using the variables τ and G which are related to those used above as follows:

$$\tau = \pm \sqrt{\eta}, \tag{7}$$

$$G = \pm \tfrac{1}{2}\xi. \tag{8}$$

Epstein adopts a different convention in regard to signs from that here used. He takes G always positive while we shall find ξ to be of the same sign as $x - x_m$. Epstein however takes τ to be of the same sign as $x - x_m$.

Epstein simplifies Eq. 6 by grouping together factors under the symbol L thus:

$$L = 2(\pi/2kT)^{3/4}m^{1/4}(ei)^{1/2}. \tag{9}$$

Equation 6 thus becomes:

$$\xi = 2L(x - x_m). \tag{10}$$

By the introduction of the new variables η and ξ the second integration

referred to previously takes the form

$$\xi = \int_0^{\eta} \frac{d\eta}{\left[\epsilon^{\eta} - 1 \pm \epsilon^{\eta} P(\sqrt{\eta}) \mp \frac{2}{\sqrt{\pi}} \sqrt{\eta} \right]^{1/2}}. \tag{11}$$

Here P represents the probability function so that

$$P(\sqrt{\eta}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\eta}} \epsilon^{-\eta^2} d\eta. \tag{12}$$

In Eq. (11) the upper or the lower signs are to be taken according as $x - x_m$ is negative or positive, respectively.

When the relation between ξ and η has been found by Eq. (11) then Eqs. (5) and (6) contain the complete solution of the problem of the potential distribution between the electrodes in terms of the current and the temperature, etc. To make practical use of these results, however, it is necessary to prepare a table of the function $\xi(\eta)$. Adams in 1913 calculated this function to about four figures while Epstein and Fry have given only rough values sometimes inaccurate to several per cent. Adams's calculations have recently been repeated by improved methods and to a higher degree of accuracy by Miss Katharine B. Blodgett. For small values of η the method given by Epstein (in which τ is used as a parameter instead of η) is much more convenient for calculating ξ than that used by Adams or Fry. For this reason values of ξ corresponding to values of τ from -0.8 to $+0.8$ have been calculated by means of Epstein's integral for intervals of 0.1 unit. These results are recorded in Table I. From these by interpolation, using Newton's method, the values of ξ corresponding to integral decimals of η have been calculated. For ordinary use it is more convenient to use a table containing ξ as a function of η rather than of τ, since η is proportional to the potentials measured from the minimum potential surface. For purposes of interpolation for small values of η or τ, however, the table having τ as a parameter is more accurate.

TABLE I.

ξ in Terms of τ.

τ.	$-\xi$.	$+\xi$.	τ.	$-\xi$.	$+\xi$.
0.0	0.00000	0.00000	0.5	0.90283	1.09092
0.1	0.19621	0.20372	0.6	1.05917	1.33009
0.2	0.38476	0.41482	0.7	1.20714	1.57591
0.3	0.56547	0.63315	0.8	1.34659	1.82830
0.4	0.73821	0.85857			

The final values of ξ as a function of η are given in Table II. These data are believed to be accurate to the number of figures given, as they have been calculated to two extra decimal places and have been checked by using both Simpson's and Weddle's rules in evaluating the integral of Eq. (11).

TABLE II.

ξ in Terms of η.

ξ is always of the same sign as $x - x_m$.

η.	$-\xi$.	$+\xi$.	η.	$-\xi$.	$+\xi$.
0.00	0.0000	0.0000	6.5	2.4990	7.1924
.05	.4281	.4657	7.0	2.5112	7.5345
.10	.5941	.6693	7.5	2.5206	7.8690
.15	.7167	.8296	8.0	2.5280	8.1963
.20	.8170	.9674	9.0	2.5382	8.8323
.25	.9028	1.0909	10	2.5444	9.4465
.30	.9785	1.2042	11	2.5481	10.0417
.35	1.0464	1.3098	12	2.5504	10.6204
.40	1.1081	1.4092	13	2.5518	11.1845
.45	1.1648	1.5035	14	2.5526	11.7355
.50	1.2173	1.5936	15	2.5531	12.2747
.6	1.3120	1.7636	16	2.5534	12.8032
.7	1.3956	1.9224	18	2.5537	13.8313
.8	1.4704	2.0725	20	2.5538	14.8260
.9	1.5380	2.2154	25	2.5539	17.1931
1.0	1.5996	2.3522	30		19.4253
1.1	1.6561	2.4839	35		21.5522
1.2	1.7081	2.6110	40		23.5939
1.4	1.8009	2.8539	45		25.5643
1.6	1.8813	3.0842	50		27.4740
1.8	1.9515	3.3040	60		31.141
2.0	2.0134	3.5151	70		34.642
2.2	2.0681	3.7187	80		38.007
2.4	2.1168	3.9158	90		41.258
2.6	2.1602	4.1071	100		44.412
2.8	2.1990	4.2934	150		59.086
3.0	2.2338	4.4750	200		72.479
3.2	2.2650	4.6524	300		96.877
3.4	2.2930	4.8261	400		119.185
3.6	2.3183	4.9963	500		140.068
3.8	2.3410	5.1634	600		159.885
4.0	2.3615	5.3274	700		178.861
4.5	2.4044	5.7259	800		197.146
5.0	2.4376	6.1098	900		214.850
5.5	2.4634	6.4811	1000	2.5539	232.054
6.0	2.4834	6.8416			

For large values of η the function ξ has been calculated by a series expansion.

The following expression for negative values of ξ is accurate to one unit in the fifth decimal place for all values of η greater than 3.0:

$$\xi = -2.55389 + \sqrt{2}\,\epsilon^{-\eta/2} - 0.0123\epsilon^{-\eta} + \frac{1}{3\sqrt{2}}\left(\sqrt{\frac{\eta}{\pi}} + 1\right)\epsilon^{-3\eta/2}. \quad (13)$$

For positive values of ξ Adams obtained the following series which is

accurate to one unit in the fourth decimal place when η is greater than 8:

$$\xi = 1.25520\eta^{3/4} + 1.66854\eta^{1/4} - 0.50880 - 0.1677\eta^{-1/4} + 0.1441\eta^{-3/4} - 0.0145\eta^{-5/4} - 0.069\eta^{-7/4} + 0.036\eta^{-9/4} + 0.083\eta^{-11/4}. \quad (14)$$

The coefficient of the first term is equal to $(2/3)\sqrt{2}\pi^{1/4}$ so that as a first approximation for large values of η we have

$$\xi = (2/3)\sqrt{2}\pi^{1/4}\eta^{3/4}. \quad (15)$$

By substituting this in Eq. (6), squaring, combining with Eq. (5) and neglecting V_m and x_m compared with V and x, we obtain

$$i = \frac{\sqrt{2}}{9\pi}\sqrt{\frac{e}{m}}\frac{V^{3/2}}{x^2}, \quad (16)$$

which is the usual three halves power law equation as derived by Child and Langmuir.

If we take into account the second term in the expansion of (14) and consider that V_m and x_m are not negligible, we obtain

$$i = \frac{\sqrt{2}}{9\pi}\sqrt{\frac{e}{m}}\frac{(V - V_m)^{3/2}}{(x - x_m)^2}\left(1 + \frac{2.66}{\sqrt{\eta}}\right). \quad (17)$$

Numerical Calculations.—In applying the foregoing results to the numerical solution of problems we proceed as follows:

By combining Eqs. (3) and (5) we obtain

$$\eta_1 = \log\frac{i_0}{i}, \quad (18)$$

where i_0 is the saturation current and i is the actual current between anode and cathode. This gives the value of η corresponding to the surface of the cathode.

Let us now place in Eqs. (5) and (9) the values of the constants $e = 4.774 \times 10^{-10}$ c.g.s. units; $k = 1.372 \times 10^{-16}$ erg/degree; $m = 9.01 \times 10^{-28}$ g.

Then if we express V in volts, we find $e/k = 11{,}600$ degrees per volt. By applying Eq. (5) first to the surface of the cathode and then to any other point we find

$$V - V_1 = T(\eta - \eta_1)/11600. \quad (19)$$

If we express i in amperes, then Eq. (9) becomes

$$L = 4.590 \times 10^5 T^{-3/4}\sqrt{i}\ \text{cm}^{-1}. \quad (20)$$

By applying Eq. (10) first to the surface of the cathode and then to another point we obtain, by subtraction,

$$2L(x - x_1) = \xi - \xi_1. \quad (21)$$

These last four equations together with the relation between ξ and η given by Table II. furnish the complete solution of the problem of the potential distribution and the magnitude of the current between parallel planes. Knowing the saturation current and choosing a value of the current i we find η_1 from Eq. (18) and L from Eq. (20). From the table we now look up the value of $-\xi$ corresponding to η_1 and call this value $-\xi_1$. If we wish to measure distances and voltages from the surface of the cathode, then V_1 and x_1 become zero. By choosing any value of x we then find ξ from Eq. (21) and from the table obtain the corresponding value of η. Eq. 19 then gives the voltage at this point.

If it is desired to use the approximation formula (17), the value of V_m is found from Eq. (19) by placing $V_1 = 0$ and $\eta_m = 0$ thus

$$V_m = -T\eta_1/11600 \tag{22}$$

in which η_1 may be found from Eq. (18).

Similarly from Eq. (21) we find

$$x_m = -\xi_1/2L, \tag{23}$$

in which $-\xi_1$ is a number (always less than 2.56) found by the table from the corresponding value of η_1, and L is obtained from Eq. (20). The quantity η according to Eq. (19) is then

$$\eta = \eta_1 + (11600V/T). \tag{24}$$

As an illustration let us take the example considered by Fry, of a surface of tungsten at 2400° K capable of giving a saturation current of 0.16 amp. per cm². The anode is a parallel surface at a distance of 0.5 cm. From Eq. (20) we find $L = 1339\sqrt{i}$. Taking the cathode as origin we then obtain [1] from Eqs. (19) and (21) for the surface of the anode at $x_2 = 0.5$,

$$V = 0.207(\eta - \eta_1), \tag{25}$$

$$\xi_2 = \xi_1 + 1339\sqrt{i}. \tag{26}$$

[1] Fry obtains equations of this form but gets the coefficients 0.3 and 1010 instead of 0.207 and 1339. These differences are due to two errors arising from confusion of different kinds of averages. The *average velocity of the emitted electrons* which Fry denotes by $\bar{v}_0$ is actually equal to $(\pi kT/2m)^{1/2}$ but if we substitute this in Fry's equations we get different equations from those obtained here. The potential change $\overline{V}_0$ is defined as that "which would give to an electron an energy equal to the *average energy* of those shot out from the cathode." This average energy as Richardson has shown is $2kT$ whereas if we substitute the value of $\bar{v}_0$ in the second equation on page 449 of Fry's paper we find that his value for the average energy is πkT. The third equation on the same page is also wrong for the exponent has the value $V'e/\pi kT$ whereas it should be $V'e/kT$. On page 450 Fry states that the value of $\overline{V}_0$ for tungsten at 2400° K is 0.3 volt. This corresponds to the value $(3/2)kT$ which is the average energy of the electrons in a given volume and not the energy of the electrons passing through a given surface $(2kT)$.

TABLE III.

Current between Parallel Plane Electrodes 0.5 cm Apart.

1	2	3	4	5	6	7	8	9	10	11
i/i_0.	i amp.	η_1.	ξ_1.	ξ_2.	η_2.	V_2 volts.	V_m volts.	x_m cm.	i/i_3.	i_4/i_3
0.001	0.00016	6.908	−2.509	14.4	19.2	2.5	−1.43	0.074	4.22	4.30
0.01	0.0016	4.605	−2.411	51.2	122.4	24.4	−0.95	0.0224	1.424	1.44
0.1	0.016	2.303	−2.094	167.3	638.6	131.6	−0.48	0.0062	1.134	1.138
1.0	0.16	0.000	−0.000	535.6	3,117.4	645.0	0.00	0.0000	1.045	1.048

Taking the current i to be a decimal fraction of the saturation current i_0 as indicated in the first two columns, η_1 is calculated by Eq. (18) (Col. 3). From this, ξ_1 (Col. 4) is found from Table II., keeping in mind that ξ_1 must be negative since at the cathode x is zero and therefore less than x_m. From Eq. (26) the value of ξ_2 is found (Col. 5) and then η_2 (Col. 6) by Table II. The voltage given in Col. 7 is that of the anode with respect to the cathode calculated by Eq. (25) from η_2 and η_1. Col. 8 gives the potential of the minimum potential point with respect to the cathode, calculated by Eq. (22), while x_m (Col. 9) is the distance of this point from the cathode calculated by Eq. (23).

Col. 10 gives data for comparing the results of these calculations with those obtainable from the ordinary three halves power law. When i and V in Eq. (16) are expressed in amperes and volts and the values of e and m are introduced, the equation takes the form

$$i_3 = 2.336 \times 10^{-6}(V_2^{3/2}/x^2) \text{ amp./cm}^2. \qquad (27)$$

We thus denote by i_3 the current *calculated* in this way from the corresponding value of voltage given in Col. 7 (V_2). The ratio i/i_3 of the actual current i (Col. 2) to that given by the three halves power law, Eq. (27), is tabulated in Col. 10.

Col. 11 illustrates the degree of accuracy of the second approximation equation (17). Let us denote by i_4 the currents calculated by Eq. (17) from the corresponding values of V_2 (Col. 7). Col. 11 contains the ratios of i_4 to i_3. The close agreement between Cols. 10 and 11 shows that the currents calculated by Eq. (17) are very nearly equal to the actual currents i. The figures in Col. 11 give directly the factor by which the currents calculated by the ordinary three halves power law should be multiplied in order to obtain the results that could be got from the better approximation of Eq. (17).

The deviations from the three halves power law indicated in Col. 10 are very much less than those calculated by Fry from the same data.

Fry states in his conclusion that the errors in current involved in the use of the three halves power law (in the example here considered) "may be close to 50 per cent for voltages as high as 40 or 50 volts." Performing calculations like those involved in Table III. shows, however, that the actual current at an anode voltage of 50 is only 27 per cent larger than calculated by the three halves power law.

The deviations are however rather large and would be very important if space charge phenomena are to be used for the determination of e/m. It must not be thought that the deviations are as large as this in the case of cylindrical electrodes. The measurements made by Dushman [1] of electron currents, limited by space charge, from a wire to a concentric cylindrical anode, are quite incompatible with deviations as great as those indicated by Table III.

Schottky [2] pointed out that the correction of the three halves power law for cylinders must be much smaller than for parallel planes. The term corresponding to x_m should vanish since it could only change the effective diameter of the cathode and this diameter does not enter the equation for cylinders. The term containing V_m should remain subtracted from the anode voltage as in the case of parallel planes.

The effect of initial velocities on the current between cylinders will be most marked near the anode where the field is weakest. The space charge at any point will be reduced approximately in the ratio $[(V/(V + V_0)]^{1/2}$, where V is the voltage at the point and V_0 is the voltage corresponding to the average initial energy of the electrons in a radial direction. Substituting this corrected space charge in the differential equation and considering that the correction is small we are enabled to obtain an approximate equation for the effect of initial velocities on the current between concentric cylinders. This equation is

$$i = \frac{2\sqrt{2}}{9}\sqrt{\frac{e}{m}}\left[V - V_m + \frac{V_0}{4}\left(\log \frac{V}{\lambda V_0}\right)^2 \right]^{3/2} \Big/ \beta^2 r, \qquad (28)$$

where i is the current per unit of length, V is the potential of the anode, V_m is the potential at the minimum potential surface (given as before by Eqs. (22) and (18)), r is the radius of the anode, β is a constant nearly equal to unity and λ is a numerical constant whose value probably lies between 1 and 2 and must be determined by experiment. Even without knowing the exact value of λ the magnitude of the corrections may be estimated from this equation. The average kinetic energy component normal to the surface among the electrons leaving a surface is kT while each component parallel to the surface is $\frac{1}{2}kT$. In the case of a small

[1] PHYSICAL REVIEW, *4*, 121 (1914).
[2] Physik. Zeitschr., *15*, 624 (1914).

wire in a large cylinder not only the normal component but the component tangent to a cross section of the cathode wire will be effective in producing radial velocity components. Therefore the average radial energy component is $(3/2)kT$,

$$V_0 = (3/2)kT/e = T/7{,}733 \text{ volts.} \tag{29}$$

If V and V_m in Eq. (28) are expressed in volts, r in cm, and i in amp. per cm, then the coefficient in Eq. (28) becomes 14.68×10^{-6}.

As an example comparable with that illustrated in Table III., let us consider a tungsten filament 0.25 mm in diameter, at 2,400° K, in the axis of a cylindrical anode, 1 cm in diameter. If the electron emission at this temperature is 0.16 amp. per cm², the saturation current i between cathode and anode is 0.0126 amp. per cm of length. According to Eq. (29) we find that $V_0 = 0.31$ volt. To find the anode voltage at which the current becomes saturated, we place $V_m = 0$ in Eq. (28) and solve for V. If we assume that $\beta = 1$ and $\lambda = 1$, we find $V = 55$ volts. The current calculated from Eq. (28) is 5 per cent greater than if calculated from the ordinary three halves power law neglecting initial velocities. This correction is several times smaller than that applying to parallel planes as given in Table III.

In order to compare the corrections at voltages of 24.4 and 131, which were used in Table III., we must carry out the calculation at higher currents so that saturation does not occur. If the filament temperature is raised to 2,540° K, the saturation current will be 0.045 amp. per cm and the saturation voltage will be 131. The current calculated by Eq. (28), taking $\lambda = 1$ and $\beta = 1$, is then 1.033 times greater than by the ordinary three halves power law whereas by Table III. the ratio is 1.134 for parallel planes. Thus the correction for initial velocities for cylinders is about one fourth as great as for planes. A similar result is obtained for an anode voltage of 24.4. We then find $V_m = -0.53$ volt, and by Eq. (28), $i = 0.004$ amp. per cm, this calculated current being 1.12 times as great as by the three halves power law, while Table III. gives 1.424 for planes. If we had taken $\lambda = 2$ instead of 1 the corrections for cylinders would have been about one fifth those for planes instead of one fourth.

Experiments are now in progress to measure accurately the relation between the voltage and the current in cylindrical anode tubes so as to test Eq. (29) and determine the number represented by λ.[1]

Case II. Condition of Equilibrium (Zero Current) without External Field (Richardson-Laue).

Consider a plane cathode with the opposing plane anode parallel to it but at an infinite distance from it, and let the potential of the anode

[1] See Note at end of paper.

be such that there is no potential gradient near the anode. Under these conditions we shall see that the anode is at an infinite negative potential so that no current flows between the electrodes. This condition may be approached in Case I. if we increase the distance between the electrodes and if we make the anode potential such that only a very small current flows. The potential distribution near the cathode then becomes close to that calculated by the method we shall now consider. However the method of Case I. involves difficulties when the current is a very minute fraction of the saturation current, for η_1 by Eq. (18) becomes very large, and thus ξ_1 (by Table II.) becomes almost exactly equal to -2.5539, while L, by Eq. (20), is close to zero. When it is attempted to calculate the potential gradient near the cathode by Eqs. (19) and (21), the Eq. (21) is found to become nearly indeterminate and therefore unsuitable for this calculation. The equations given below are then more convenient.

Since we are dealing with equilibrium conditions, the distribution of electrons is given by Boltzmann's equation

$$\rho = \rho_1 \epsilon^{\frac{Ve}{kT}}, \tag{30}$$

where ρ_1 is the electron space charge at the surface of the cathode, and ρ is the space charge at any point whose potential with respect to the cathode is V. Substituting this value of ρ in (4) (Poisson's equation), integrating twice, and imposing the condition that $dV/dx = 0$ when $x = \infty$, we find

$$V = -\frac{kT}{e} \log \left[\frac{2\pi e \rho_1}{kT} (x - x_0)^2 \right], \tag{31}$$

where x_0 is an integration constant. This is essentially the equation obtained by Laue.

The space charge ρ_1 may be calculated from Eq. (2) by integration by considering that dN_s/v is the number of electrons per unit volume having velocity components between v and $(v + dv)$ and by noting that the saturation current i_0 is equal to eN_s. Thus we find

$$\rho_1 = i_0 (2\pi m/kT)^{1/2}. \tag{32}$$

We shall find it more convenient to use the same nomenclature as in Case I. Let us denote by L_0 the value of L calculated from Eqs. (9) or (20) by placing $i = i_0$. Using this we can eliminate i_0 from (32) and obtain

$$\rho_1 = (kT/\pi e) L_0^2. \tag{33}$$

The integration constant x_0 in Eq. (31) can be eliminated if we measure x and V from the surface of the cathode, for then $V = 0$ when $x = 0$.

Substitution of the value of ρ_1 from Eq. (33) into Eq. (31) gives

$$\sqrt{2}L_0x = \epsilon^{-\frac{Ve}{2kT}} - 1. \tag{34}$$

From Eqs. (30) and (34) we may eliminate the exponential term and obtain a convenient expression for the space charge at any point:

$$\rho = \rho_1/(\sqrt{2}\,L_0x + 1)^2. \tag{35}$$

When x is very large compared to $1/L_0$ so that the term 1 in the denominator may be neglected, Eqs. (33) and (35) can be combined to give the simple relation

$$\rho = kT/2\pi ex^2\,. \tag{36}$$

Thus, except very close to the cathode, *the electron density is independent of the material of the cathode, is proportional to the temperature and is inversely proportional to the square of the distance from the cathode.*

Eq. (34) can also be obtained as a limit for Case I. for large negative values of ξ. Taking only the first variable term in the expansion (13) we find

$$\xi - \xi_1 = \sqrt{2}(\epsilon^{-\eta/2} - \epsilon^{-\eta_1/2}). \tag{38}$$

From Eqs. (9) and (18)

$$L^2 = L_0^2\epsilon^{-\eta_1}. \tag{39}$$

Placing $x_1 = 0$ in Eq. (21) and combining with Eqs. (38) and (39) we find

$$\sqrt{2}\,L_0x = \epsilon^{-(\eta-\eta_1)/2} - 1,$$

which becomes identical with (34) when we substitute the value of $\eta - \eta_1$ from Eq. (5).

The potential gradient dV/dx at the surface of the cathode, which we may call X_1, is found by differentiating (34) to be

$$X_1 = -2\sqrt{2}\,\frac{kT}{e}\,L_0. \tag{40}$$

As an illustration of the use of these equations, let us consider a large flat tungsten surface at 2,400° K in a large glass bulb. The conditions then correspond closely to those assumed in Case II. With Fry we may assume the saturation current obtainable from the tungsten to be $i_0 = 0.16$ ampere per cm^2. Substituting the values of e, m, and L_0 in Eq. (40) as in Eqs. (19) and (20) we find

$$X_1 = -111.92T^{1/4}\sqrt{i_0}\text{ volts per cm}, \tag{41}$$

which in the example considered gives $X_1 = -314$ volts per cm. This is the electric field at the surface of the cathode due to the space charge of the electron atmosphere near the cathode.

Placing $T = 2{,}400$ and $i_0 = 0.16$ in Eq. (20) we find $L_0 = 535$ per cm, which enables us to solve Eq. (34) for V, giving

$$V = -0.95 \log_{10} (758x + 1) \text{ volts.}$$

The voltages calculated from this for several values of x are given in Table IV.

The space charge ρ_1 at the surface of the cathode is found from Eq. (32) to be

$$\rho_1 = 19260 i_0 / \sqrt{T} \text{ e.s.u. per cm}^3,$$

TABLE IV.

Potential Distribution and Electron Density near a Tungsten Cathode at 2,400° K, without External Field.

Distance from Cathode, Cm.	Potential Volts.	Electrons per Cm³.
0	0.00	1.32×10^{11}
0.0001	−0.03	1.14×10^{11}
0.001	−0.23	4.26×10^{10}
0.01	−0.88	1.79×10^{9}
0.1	−1.79	2.23×10^{7}
1	−2.74	2.28×10^{5}
10	−3.69	2.29×10^{3}

when i_0 is expressed in amperes per cm². Inserting the values of i_0 and T gives $\rho_1 = 62.8$ e.s.u. per cm³, or 1.315×10^{11} electrons per cm³. The electron densities for other distances from the cathode calculated by Eq. (35) are given in the last column of Table IV.

CASE III. CONDITION OF EQUILIBRIUM (ZERO CURRENT) WITH RETARDING FIELDS.

Consider an electron-emitting plane surface (cathode) and a second parallel plane electrode (anode) at a great distance. Let the anode be at such a large negative potential that the electrons are pressed back against the cathode so that only a negligible number pass beyond a certain distance from the cathode. Beyond this distance there is a uniform potential gradient which we may represent by X_∞. Laue has given the solution for the potential distribution in the form

$$-\frac{Ve}{kT} = \log \left[\sinh \frac{X_\infty e(x - x_0)}{2kT} \right]^2 - \log \frac{X_\infty^2 e}{8\pi kT\rho_1} . \tag{42}$$

If we put

$$X_1 = -(8\pi kT\rho_1/e)^{1/2}, \tag{43}$$

Eq. (42) can be transformed into the more convenient form

$$\sinh^{-1}\left(\frac{X_\infty}{X_1}\epsilon^{-\frac{Ve}{2kT}}\right) - \sinh^{-1}\left(\frac{X_\infty}{X_1}\right) = -\frac{X_\infty ex}{2kT}. \qquad (44)$$

It should be noted that the X_1 defined by Eq. (43) is the same as that given by Eq. (40), as can be readily proved by substituting the value of L_0 from Eq. (33) into Eq. (40). For values of X_∞ so small that only the first term in the series expansion of $\sinh^{-1}$ is needed, the above equation reduces to Eq. (34) of Case II. after the value of X_1 from Eq. (40) is introduced.

By differentiation of (44), the potential gradient dV/dx at the surface of the cathode, which we may call X_s, is found to be

$$X_s = -\sqrt{X_1^2 + X_\infty^2}.$$

These equations lend themselves readily to numerical calculations. The value of X_1 is found from Eq. (41) and it is then only necessary to place $e/k = 11{,}600$ degrees per volt in Eq. (44), in order to calculate the potential distribution.

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SCHENECTADY, N. Y.,
November 14, 1922.

[1] *Note* added Jan. 8, 1923. Recent mathematical analysis shows that an error was made in the calculation of β in the writer's paper on space charge (PHYSICAL REV., *2*, 450, 1913, and Physik. Zeitschr., *15*, 348, 1914), in which β was found almost exactly equal to unity for all cases where the radius of the anode (r) is more than twenty times that of the cathode (a). The following table gives the new values of β^2 for several values of r/a.

r/a.	β^2.	r/a.	β^2.	r/a.	β^2.
7.0	0.8870	16.0	1.0513	121.5	1.0722
8.0	0.9252	20.09	1.0718	221.4	1.0531
9.0	0.9547	29.96	1.0908	735.1	1.0225
10.0	0.9782	44.70	1.0945	2,440	1.0064
12.0	1.0122	66.69	1.0889	22,026	0.9990

The function β approaches unity in a series of oscillations of decreasing amplitude. The agreement observed by Dushman between experiment and the ordinary space charge equation for cylinders (taking β equal to unity) was due to two compensating errors. The corrections corresponding to the terms containing V_∞ and V_0 in Eq. (28) amount to 8.1 per cent at 35 volts on the anode, 5.0 per cent at 75 volts and 3.5 per cent at 130 volts (assuming $\lambda = 1$). The correction due to β^2, which should have been put equal to 1.079 instead of unity, is 7.3 per cent in the opposite direction. In the range from 35 to 90 volts the currents calculated with these two corrections agree with Dushman's observations within the experimental error. At 130 volts the observed current was about 4 per cent higher than that calculated. This may have been due to a trace of ionization of residual gas.

THE INFLUENCE OF THE EARTH'S POTENTIAL-GRADIENT UPON MEASUREMENTS OF THE MEAN IONIC DENSITY OF THE ATMOSPHERE BY THE EBERT ION-COUNTER.[1]

By James F. Mackell.

Abstract.

Effect of potential-gradient upon measurements with an Ebert ion-counter, for 35 to 115 volts/meter.—It was pointed out by Swann in 1914 that because of the negative charge induced at the upper end of the earthed outer shell by the atmospheric gradient, the density of negative ions as measured must be considerably less than the true value, while the density of positive ions is unaffected. Because of the wide use of the instrument for routine observations, it is important to settle the matter experimentally. Two identical ion-counters of the ordinary type were mounted two meters apart with axes normal to the outside wall of the laboratory and one meter from it. The true ion density N was determined by readings with the ion-counter shielded from the gradient by a large wire cage lowered over it; the differential effect of the gradient, ΔN, by readings made with shielded and unshielded counters connected to opposite quadrants of the electrometer; the potential-gradient, by using a polonium plate connected to an electroscope. It was found that for the negative ions, the relative error $\Delta N/N$ increased linearly with the potential-gradient from 17 per cent for 35 volts/meter to 27 per cent for 115 volts/meter, whereas for the positive ions the average error was zero within the limits of accuracy of the experiment. These results are in agreement with the theory.

Introduction.

One of the most common methods of determining the ionic density of the atmosphere is by the use of the so-called Ebert ion-counter, designed by Professor Ebert some twenty years ago. The general method employed is as follows: Air is drawn into a cylindrical condenser, the central member of which is charged to a potential of sufficient magnitude to enable it to attract to itself all the ions of opposite sign which enter the tube, the outer system being earthed. The charge upon the central system then falls off and the rate of discharge may be noted by the effect produced upon a charged electroscope or electrometer system. In this way the ionic density for the positive and negative ions may be determined if the rate of air flow and the capacity of the apparatus are known.

It is of course obvious that with the usual sign of the earth's potential-

[1] Presented at the Chicago meeting of the American Physical Society, December 28–30, 1920.

gradient, a negative charge will appear at the top of the ion-counter where the air is drawn into the tube. This may be expected to have an effect upon the number of ions entering the tube in unit time. Professor W. F. G. Swann [1] has shown from mathematical theory that the effect is to reduce the number of negative ions entering the instrument, while in the case of the positive ions, there is no error, since just as many ions are caught by the charge on the top of the instrument as corresponds to the excess of positive ions which are drawn to the system on account of the induced charge. He shows that if ΔN be the error in the number of ions which enter per second, and N the number which would have entered per second in the absence of the charge Q on the inside of the top of the apparatus, then

$$\frac{\Delta N}{N} = \frac{4\pi Q v}{\text{Flow of air per second}},$$

where v is the specific velocity of the ions.

In the same paper Professor Swann describes a series of four experiments in which he made a direct measurement of Q for known values of the potential-gradient. His calculations show that in the case of the instrument used, the error should be of the order of 26 per cent for a positive potential-gradient of 70 volts per meter, and he naturally concludes that an error of this size might play an important rôle in accounting for the difference usually found between the ionic densities of the positive and negative ions. It is also apparent that in so far as this effect might be expected to become greater with increase of the potential-gradient, the error in the case of high atmospheric potentials would become enormous. It is obviously of importance that a matter of this kind affecting measurements made by observatories for periods extending over many years should receive the most searching inquiry.

The ordinary Ebert ion-counter at present on the market is provided with a conical umbrella-like attachment (the cap), which is fixed to the upper end. The original purpose of this cap was to keep insects and other foreign matter out of the apparatus. It, however, plays an important part in influencing the conditions as regards the induced charge on the open end of the apparatus, not, it is true, as drastic a rôle as might at first sight appear; for, although it shields the open end from the potential-gradient, it itself takes the place of this open end and acquires a charge which has an influence on the measurements. Swann's theory applies to an opening of any shape, even to one constituted by the mouth of the cap, provided that the appropriate experimentally measured value of Q be utilized. Since in his experiment the purpose was merely to

[1] Terrestrial Magnetism and Atmospheric Electricity, Vol. 19, p. 205, 1914.

illustrate the theory by obtaining the order of magnitude of the correction, and since, moreover, most of the work up to that time had been done with instruments without a cap, he for simplicity used an instrument of this type.

In 1916 E. H. Nichols [1] published the results of some investigations carried out at Kew Observatory in order to test the conclusions reached by Swann. He analyzed the data at Kew taken over a period of several years, setting the measured ionic densities for the negative ions at low potentials against the corresponding densities for high potentials. He also carried out a set of experiments in which an attempt was made to eliminate the inductive effect by giving the outer system a positive charge after insulating it. Considerable effort was also made by Nichols to investigate the equipotential surfaces in the neighborhood of the apparatus and this is discussed at some length in his paper. A summary of Nichols's conclusions is as follows:

1. An examination of the Kew data in the manner stated above does not show that an appreciable inductive effect is apparent.

2. Experiments made with two Ebert electrometers to find the effect on the recorded negative charge of partly neutralizing the induced charge by means of a battery showed no effect to substantiate Swann's theory.

3. Other experiments investigating the equipotential lines near the instruments, the wind velocity and the stream lines of the air motion in the vicinity of the opening, failed to support Swann's theory in the case where the cap was used, whereas, in the case where the cap was removed, the results indicated that the theory was sound, so that the theory seems to apply only to this case.

In Swann's [2] reply to this criticism as regards the cap, he points out that his theory is applicable to any form of opening, even to one in which the cap takes part in determining the form of the opening, but that the experimentally measured value of Q for the opening must be determined in order that his formula may be applied, and it is not legitimate to draw *quantitative* conclusions as to what the theory has to say as regards an instrument with a cap from data obtained by using an instrument without one. He points out, however, that some of Nichols's data intended as evidence that the error is small in the case of the instrument with a cap actually support the reverse conclusion when interpreted as it appears they should be.

In view of the above considerations and of the very large amount of work that has been done, and is being done, with instruments of this

[1] Terr. Mag. and Atmos. Elec., Vol. 21, p. 87, 1916.

[2] Terr. Mag. and Atmos. Elec., Vol. 21, p. 99, 1916.

type, it seemed altogether fitting that a direct set of measurements be made with a capped instrument to consider the point in question. With this idea in view, the present work was commenced in the spring of 1920 at the suggestion of, and under the direction of, Professor W. F. G. Swann at the University of Minnesota.

While measurements were being made and while data organization was in progress, a communication was received from Dr. Harold Norinder of the University of Upsala, Upsala, Sweden, to the effect that a similar piece of work was being carried on at that place, and the data produced indicated a substantiation of Swann's theory. While the present paper was being prepared and after the material herewith had been collected and organized, Dr. Norinder's paper appeared in final form. Thus a valuable means is offered the author for comparative purposes.

Dr. Norinder [1] carried out his experiments in two distinct parts, one set in the laboratory, in which he produced an artificial potential-gradient between parallel plates, and another set in the open air. In both instances he found, as predicted by Swann's theory, a decided effect in the case of the negative ions but no appreciable effect in the case of the positive ions. His values for the apparent error range from four per cent to fifty-eight per cent, and, as far as they go, show a tendency to follow the potential-gradient, although the data are not sufficient to afford conclusive evidence on the latter point.

Experimental Procedure in the Present Work.

Two ion-counters of the Ebert type were made and care was taken to have them as nearly exact duplicates as possible. The general shape and proportions of these instruments are shown in Fig. 1, which also shows the general arrangement of the apparatus. The dimensions were approximately the same as those of the Ebert type commonly sold by instrument makers. Chamber E was a cylindrical metal box 10 centimeters in diameter, while the inner cylinder, C, was 57 centimeters long and 0.5 centimeter in diameter. The inner cylinder, C, was set into hard rubber insulation and surrounded by a guard ring, F, set in hard rubber, which in turn was bound by a brass ring soldered to the lower part of the chamber, E.

The two instruments were mounted about two meters apart on a board about five meters long with cross arms two meters long so as to enable a hemispherical cage to be placed over either instrument, as shown in Fig. 1. The cage was two meters in diameter and was made of wire netting with a mesh of about two centimeters. As it was found in-

[1] Arkiv fur Mathematik, Astronomi, and Fysik, Bd. 15, No. 2.

convenient to mount the instruments over the ground, and as it obviously makes no difference whether the ion-counter cylinders occupy vertical or horizontal positions, so long as their lengths are parallel to the field, the board was mounted vertically in the open air at a distance of one meter from the east wall of the Physics Building and about midway

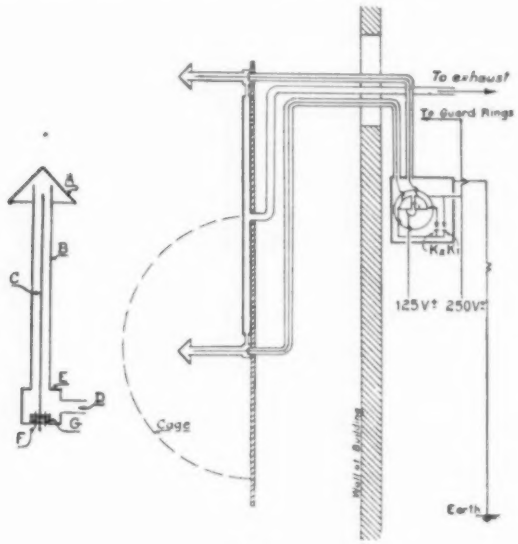

Fig. 1.

between the ground and the roof. The instruments were thus in a vertical position with respect to the wall of the building, about eight meters from the ground, and the same distance from the roof. Exhaust pipes leading to the lower part, *E*, of either instrument led to a suction fan inside the building, and the velocity of the air through a known cross section of the instrument, was observed at intervals by means of an anemometer. This was found to be fairly constant and of the order of two meters per second. The wire cage was supported by a truss from the roof and could be placed over either instrument at will by a system of pulleys. This cage was earthed by a wire leading to a water pipe as were the outer parts of the ion-counters. The inner systems were insulated and protected by guard rings, which were kept at the same potentials as the inner systems themselves.

Conducting wires shielded in metal pipes led to an electrometer on the inside of the building. The electrometer and keys were of course shielded in a metal case kept at zero potential. The needle of the electrometer was kept at a potential of $\pm$ 125 volts, while the case of the electrometer was maintained at a potential of $\pm$ 250 volts. One pair

of quadrants was connected to one inner system, and the other pair to the other inner system, as shown in Fig. 1. By means of the key, K_1, the quadrants were connected to the case and were charged to a potential of 250 volts. In carrying out an experiment, the cage being absent from both instruments, K_1 was opened thus separating the quadrants from the case and leaving a potential of $\pm$ 250 volts upon the inner systems and the quadrants. K_2 was then opened, separating the quadrants. The fan being in operation, an approximate balance was established by arranging the caps so that the respective air flows into the instruments were such as to result in a very small rate of deflection of the electrometer needle. This balance was made perfect to within three or four per cent of the effect of the potential-gradient which was under investigation, and was therefore within the range of probable experimental error.

The cage was placed over first one of the instruments and then the other. In the case of the negative ions, the application of the cage resulted in a considerable rate of deflection, ΔR, of such a sign as to indicate an increase in the rate at which ions were being received by the instrument. In the case of the positive ions, no appreciable change was produced by the cage. Readings were always taken over a fixed range of the scale, and the whole insulated system was brought to a potential of $\pm$ 250 volts after each observation so as to prevent it from ever departing far from its initial value.

For the purpose of expressing the error in the measured ionic content, it was necessary to perform experiments in which a measurement was made of R, the total rate of alteration of potential of one instrument when the other was earthed. This was done with first one, and then the other, earthed, both being unshielded in each case. It was further necessary to determine reduction factors to reduce the rates of deflection, ΔR and R, to comparable units, since it would not be quite correct to assume that the capacity of one of the electrometer systems was the same when the other was insulated as when it was earthed.

Since, in the main experiments, the procedure was adopted of measuring the time for the spot of light to move over a fixed range of the scale, which was the same for all experiments, all that was necessary was to determine, for each case, the quantities of electricity corresponding to this deflection. In order to do this an experiment was performed as follows: The needle of the electrometer system was kept at a potential of 125 volts and the electrometer case at zero potential. The inner member of a Gerdien condenser was then connected to one of the quadrants, the outer member being connected to a potentiometer system in such a manner that its potential could be varied.

Suppose it was desired to determine the quantities corresponding to the cases where both electrometer quadrants were insulated: The procedure was to disconnect the quadrants and vary the potential applied to the condenser until the standard deflection, 20 centimeters, was obtained. Then if C represents the capacity of the condenser, V the potential applied to the condenser, and v the rise of potential of the electrometer, the quantity, q_1, of electricity corresponding to the rise, v, is given by $C(V - v)$. Repetition of the experiment with the condenser connected first to one quadrant system and then to the other, the quadrant not connected to the condenser being in each case earthed, gave the values, q_2 and q_3, corresponding to these cases in the main experiments. Since only the ratios of the q's are required, the absolute value of C need not be known. However, the absolute values of the q's were obtained as follows:

Both systems insulated	$q_1 = 0.0400$
A insulated, B earthed	$q_2 = 0.0411$
B insulated, A earthed	$q_3 = 0.0417$

The expressions for the ratio of the errors resulting from the potential-gradient to the measured ionic content are given by

$$\frac{q_1/t_1}{q_2/t_2} \quad \text{and} \quad \frac{q_1/t_1}{q_3/t_3}$$

for A and B respectively, t_1, t_2, and t_3 being the respective times for the spot of light to move over the fixed range of the scale. The error in the above case is estimated on the uncorrected values of the quantities. If calculated from the corrected values, we have for the ratios of the errors ΔN to the corrected value N,

$$\left(\frac{\Delta N}{N}\right)_A = 1 \Big/ \left(1 + \frac{q_2 t_1}{q_1 t_2}\right) \quad \text{and} \quad \left(\frac{\Delta N}{N}\right)_B = 1 \Big/ \left(1 + \frac{q_3 t_1}{q_1 t_3}\right),$$

which are the values recorded in the tables.

The error introduced by the cage owing to the induced charge which will obviously appear upon it has been investigated by Professor Swann [1] and has been shown to be of comparatively small order of magnitude, amounting to about four per cent for a wind velocity of one meter per second. It may be observed, moreover, that if the error were at all appreciable, it would only serve to make the results shown by the present experiments too small rather than too large in percentage of error due to inductive action.

[1] Terr. Mag. and Atmos. Elec., Vol. 19, No. 4, pp. 210–212.

The effective value of the potential gradient influencing the apparatus was measured at the time of each observation by means of a polonium collector fastened at the center of a wire supported vertically with respect to the ground, one and one half meters from the wall of the building and about five meters to the side of the apparatus. The wire led to a calibrated electroscope, which was read at intervals during each observation, and a mean value was taken in each case. Since the electroscope was not sensitive at values less than 30 volts per meter, no observations were obtained for potential-gradients less than this.

Discussion of Results Obtained.

The results obtained may be readily seen by a study of Tables I., II., and III., and the accompanying curves, Figs. 2, 3, and 4. The potential-gradient values given in Table I. for the negative ions are grouped by tens, *i.e.*, 30 to 39, 40 to 49, etc., volts per meter. It may also be pointed out in this connection that the values given for the measured potential gradient are the means of at least two readings taken during each of the observations. In some of the cases, of course, the gradient varied phenomenally during a reading, but in the most drastic cases of this sort the readings were discarded because of the large probable error in the mean. The values recorded here are therefore those in which the fluctuations were small.

The days picked for taking observations were in the main fine clear days on which the humidity conditions were nearly perfect for such work. It was found that in mid-summer the high relative humidity and other objectionable features introduced errors so great that no reliability could be placed upon the data. For each of the ranges of ten volts per meter of the measured potential-gradient, enough readings were secured to obtain fairly accurate means for all the days. The readings were taken in the main between the hours of 9:30 A.M. and 6 P.M. of each day, and in most cases during the greater part of this interval, as it was desired to make as great a use as possible of the good days.

For the sake of comparison with results obtained for the ionic content of the air per unit volume by investigators in the past, the absolute values of N_+ and N_- obtained with the respective instruments shielded are included in the tables. In calculating these, the velocity as obtained by an anemometer over a known cross section was used. This was measured only once for each set of experiments, as it was experimentally verified that this value did not vary more than ten per cent during all the observations. The mean value was found to be 2.0 meters per second.

A glance at the tables will show that the values for $\Delta N_-/N_-$ have a

TABLE I.

Data for Negative Ions.

Pot. Grad. volt/meter.	Date 1920, October.	System *A* under Test.					System *B* under Test.				
		t_1.	t_2.	N.	ΔN.	$100 \times \frac{\Delta N}{N}$.	t_1.	t_2.	N.	ΔN.	$100 \times \frac{\Delta N}{N}$.
30 to 39.....	20–23	412	80				429	125			
	25–30	383	70				395	90			
	30–5	370	75				380	88			
	1–6 [1]	375	78				401	78			
	Means	385	76	952	153	16.1	401	95	798	148	18.6
40 to 49.....	20–23	377	93				423	112			
	25–30	378	82				383	88			
	30–5	360	70				365	85			
	1–6	368	86				365	83			
	Means	371	83	890	159	17.9	384	92	822	154	18.7
50 to 59.....	20–23	378	88				365	132			
	25–30	313	82				383	90			
	30–5	353	77				355	88			
	1–6	341	88				368	102			
	Means	346	84	896	171	19.1	368	103	763	162	21.2
60 to 69.....	20–23	333	83				360	120			
	25–30	311	80				345	76			
	30–5	350	70				275	86			
	1–6	321	84				368	76			
	Means	329	79	947	179	18.9	337	90	861	176	20.4
70 to 79.....	20–23	304	90				295	122			
	25–30	289	82				305	88			
	30–5	305	82				320	92			
	1–6	300	87				302	83			
	Means	300	85	909	196	21.6	306	96	838	194	23.2
80 to 89.....	20–23	291	88				290	125			
	25–30	263	76				300	80			
	30–5	247	75				300	88			
	1–6	270	84				271	71			
	Means	268	81	970	220	22.7	290	91	886	205	23.2
90 to 99.....	20–23	264	87				282	125			
	25–30	251	80				256	85			
	30–5	249	70				270	93			
	1–6	264	79				263	79			
	Means	257	79	1,000	230	23.0	268	96	866	222	25.6
100 to 109....	20–23	259	82				235	115			
	25–30	233	80				249	78			
	30–5	270	72				200	82			
	1–6	224	85				215	82			
	Means	246	80	1,000	240	24.0	225	89	958	264	27.6
110 to 120....	20–23	248	97				205	115			
	25–30	225	76				240	80			
	30–5	246	74				235	85			
	1–6	215	92				220	75			
	Means	234	85	965	252	26.1	225	89	958	264	27.6

[1] The dates 1, 5, 6 refer to November. Each observation above recorded is itself the mean of a two or more

TABLE II.

Data for Positive Ions.

The values indicate means of two or more observations. The sign ∞ is used to denote a time so long that the rate of deflection may be regarded as negligible.[1] The minus sign against certain values of t, and the corresponding values of ΔN is to be taken as indicating an apparent *decrease* in measured ionic content as result of applying the cage. All the values of ΔN in this table are negligible within the limits of error of the experiments.

Pot. Grad. volt/meter.	Date 1920, November.	System *A* under Test.					System *B* under Test.				
		t_1.	t_2.	N.	ΔN.	$100 \times \frac{\Delta N}{N}$.	t_1.	t_2.	N.	ΔN.	$100 \times \frac{\Delta N}{N}$.
30 to 50...	8	∞	52				−11,250	51			
	15	∞	56				∞	50			
	Means	∞	54	1,090	0	0.0	−11,250	50.5	1,170	−5	−0.4
51 to 70...	8	13,200	54				∞	55			
	15	9,850	50				∞	55			
	Means	11,520	52	1,130	5	0.4	∞	55	1,080	0	0.0
71 to 90...	8	−14,750	53				13,250	52			
	15	∞	53				∞	56			
	Means	−14,750	53	1,110	−4	−0.4	13,250	54	1,100	4	0.4
91 to 110..	8	14,420	48				9,780	50			
	15	15,800	50				10,500	60			
	Means	15,110	49	1,200	4	0.3	10,140	55	1,090	6	0.6
111 to 130.	8	∞	54				∞	54			
	15	− 8,640	50				∞	58			
	Means	− 8,640	52	1,120	−7	−0.6	∞	56	1,060	0	0.0

TABLE III.

Mean Values for All Observations.

Pot. Grad. volt/meter.	N_-.	ΔN_-.	$100 \times \Delta N_-/N_-$.	N_+.	ΔN_+.	$100 \times \Delta N_+/N_+$.	$\frac{(N-\Delta N)_+}{(N-\Delta N)_-}$.	$\frac{N_+}{N_-}$.
30 to 39.....	870	150	17.2	1,130	−2	−0.2	1.59	1.31
40 to 49.....	860	156	18.1					
50 to 59.....	830	166	20.0	1,100	2	0.2	1.59	1.27
60 to 69.....	900	178	19.8					
70 to 79.....	870	195	22.4	1,110	0	0.0	1.59	1.23
80 to 89.....	930	212	22.8					
90 to 99.....	930	226	24.3	1,140	5	0.4	1.59	1.19
100 to 109....	980	252	25.7					
110 to 120....	960	258	26.9	1,090	−3	−0.3	1.56	1.14

[1] When, of two values of t, to be averaged, one of the values was infinite, the other was used by itself in place of a mean. On account of the very large values for t_1 in Table II., the corresponding readings were taken for only one tenth of the normal scale deflection, the results given in the table being then obtained by multiplying by ten.

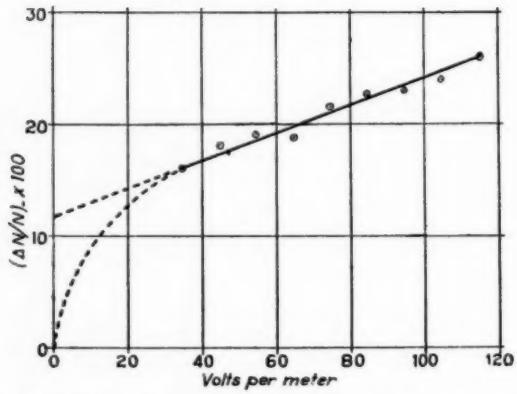

Fig. 2. Variation of percentage error with potential-gradient for System *A*.

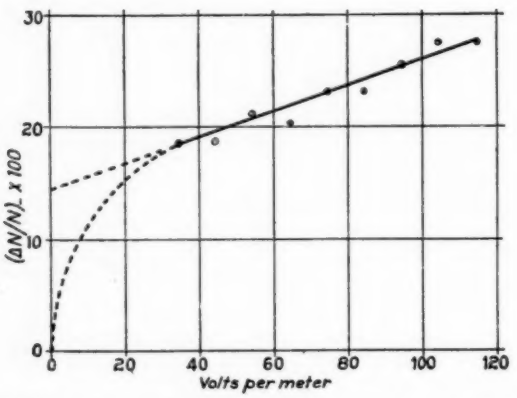

Fig. 3. Variation of percentage error with potential-gradient for System *B*.

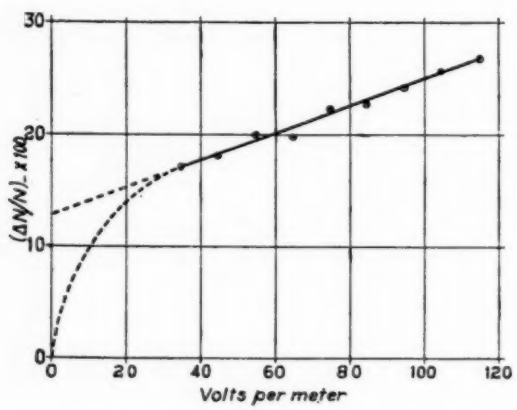

Fig. 4. Variation of percentage error with potential-gradient. Mean for Systems *A* and *B*.

decided tendency to follow the potential-gradient, the mean values ranging from 17.2 per cent to 26.9 per cent between the measured values of 30 volts per meter and 120 volts per meter of potential-gradient. The values for the lower instrument are slightly higher than the corresponding values for the upper system, but in general they follow the same course. The fact that the two instruments were not located at the same spot and that the wall of the building can not be considered absolutely plane may help to explain this small discrepancy. It must also be admitted that this small difference is none too great to be attributed to experimental error, but the fact that it is always in the same direction precludes this explanation as likely.

With regard to the first objection cited above as offered by Nichols, which was based upon the fact that the mean values for the positive and negative ionic content did not show any marked variation with the potential-gradient, such as might be expected if the gradient affected N_- and not N_+, it will be observed that the ionic content examined by Nichols was the ratio $(N - \Delta N)_+/(N - \Delta N)_-$. A glance at the table of mean values (Table III.) shows that this quantity is practically independent of the gradient as Nichols found; but, that this does not carry with it the conclusion that the values of ΔN_- are negligible is borne out by the same set of observations. The solution of the apparent paradox is contained in the fact, which becomes here experimentally demonstrated, that N_+/N_- appears itself to be a function of the potential gradient.

The fact that the curves in all cases show a very decided rise from the origin to the first observed gradient point, while from that point on they seem to follow a comparatively straight line may need a word of explanation. This may be due to the manner in which the values for the potential-gradient were obtained. The observed values for the potential-gradient are of course the differences of potential between the polonium collector one and one half meters from the wall of the building, and the case of the electroscope which was earthed to a gas pipe. These values were assumed to be the same as those at the capped ends of the ion-counters, which amounts to the same thing as assuming that the wall of the building was at the same potential as the gas pipe. One might assume from the nature of the data obtained that the wall of the building had a potential greater than this, and, working on this assumption, the slope of the curves should be used for determining the correction values in so far as they relate to the theory outlined above, although the whole error is of course that indicated by the experiment. Another explanation has been suggested by Professor Swann in the paper immediately following.

Summary of Conclusions.

1. The data obtained show that a shield, such as the cage used in this experiment, over the ion-counter does affect the results.

2. The effect for the negative ions is in the right direction and of great enough order of magnitude to substantiate Swann's conclusions. Moreover, it appears that the cap on the instrument does not eradicate the error.

3. The values of $\Delta N_+/N_+$ are practically zero, which is in accordance with the theory.

4. The mean values for $(N - \Delta N)_+/(N - \Delta N)_-$ are substantially the same for all values of the potential-gradient, which is in agreement with the data submitted by Nichols, and mentioned before in this paper, but the values for N_+/N_- vary, which leads one to infer that this latter quantity is itself a function of the potential-gradient.

In closing, the author takes pleasure in acknowledging the assistance of Professor W. F. G. Swann, who suggested the problem and method of procedure, and who was at all times ready to help with valuable suggestions for overcoming the many difficulties encountered. In designing the apparatus, the assistance of Mr. C. H. Dane, mechanician, is greatly appreciated.

Department of Physics,
University of Minnesota,
September 27, 1922.

THE THEORY OF THE ACTION OF THE EARTH'S POTENTIAL-GRADIENT IN MEASUREMENTS WITH THE EBERT ION-COUNTER.

By W. F. G. Swann.

Abstract.

Theory of effect of potential gradient on measurements with the Ebert ion-counter.—In a former paper, the theory of the action of the negative charge induced on the ion-counter by the normal atmospheric potential gradient in causing error in the measurements of the negative ion-density was worked out for the case where the induced charge is insufficient to prevent the entry of ions at any point. The theory led to a linear relation between the reduction in measured negative ionic density and the potential-gradient. The *simple theory is now extended*, to take account of the action of the induced charge in not only changing the speed of the ions but in reducing the area of the surface through which negative ions can enter the instrument. Assuming a simple ring distribution of the induced charge, an expression for the variation of percentage error or reduction in the measured negative ionic density with potential-gradient, is derived, and shows a non-linear law, in general agreement with the observations of Mackell. In measurements of positive ion-density, the extended theory, like the simple one, indicates that no error is to be expected, for a potential-gradient of normal sign, a conclusion confirmed by recent results of Mackell and of Norinder.

IN the paper immediately preceding this one in the present issue of the Physical Review, Mr. J. F. Mackell has described experiments demonstrating the existence of a large error in measurements of the negative atmospheric ionic-density with the Ebert ion-counter, as a result of the action of the potential gradient. This result was predicted by the present writer in a former paper,[1] in which it was shown that the error in question was proportional to the potential-gradient. While Mr. Mackell's results could be taken as showing, within the limits of accuracy of the experiments, a linear relation between the error and the potential-gradient for the range of potential-gradients investigated, the straight line, when produced back, does not pass through the origin. This result indicates that the true relation must really depart from an exact linear one, and the object of the present paper is to account for this phenomenon.

In the simple theory formerly deduced, it was assumed that the field due to the induced charge on the instrument was nowhere so great as to completely prevent the entrance of ions with the air current, its action

[1] Terrestrial Magnetism and Atmospheric Electricity, Vol. 19, p. 205, 1914.

being essentially that of reducing the velocity with which the ions enter, and so the number entering per second. It appeared that, while the field altered the velocities of the ions, it did not alter their density, and the ratio of the alteration ΔN in the number of negative ions entering the instrument per second to N, the total number which would have entered per second in the absence of the potential-gradient, was

$$\frac{\Delta N}{N} = \frac{v}{Q}\int\int FdS,$$

where v is the mobility of the negative ion, Q the volume of air entering the instrument per second, and the integral of FdS represents the integral of the normal component of the electric field, due to the induced charge, taken over any surface which forms a cap to the open end of the instrument. The integral is of course proportional to the charge induced on the inside of the open end, and so to the potential gradient.

Let us confine our attention to a cylindrical open end, and to a case where the velocity of the air is uniform. It is obvious that if the induced charge is sufficiently great, there will be some ring, such as that represented in section by RR, Fig. 1, at which the velocity due to the field

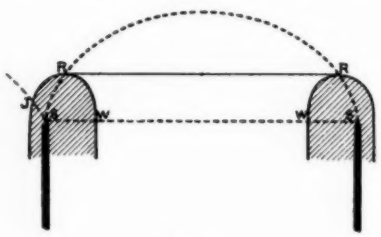

Fig. 1.

will be equal to the velocity of the air. There will be, in fact, a region such as that represented in section by the shaded region, whose boundary is determined by the condition that the velocity due to the electric field at a point of the boundary is equal to the component of the velocity of the air normal to the boundary, and no ions will be able to enter this region. The normal flux of ions per second through the cross-section represented by WW is reduced by the potential-gradient to the extent of $nv\int\int FdS$, where n is the normal number of ions per c.c., and where the integral is taken over the area of the circle of diameter WW. But the reduction of the flux of ions through the region between the circle WW and the open end SS is less than nv times the corresponding integral taken over the annulus represented in section by SW. For, the integral taken over the annulus SW is equal to the integral taken over the surface

represented in section by *JRW*, where *J* is the point at which the line of force from *S* (the top of the inside of the tube) cuts the section *JRW*. This latter integral, multiplied by nv, is, however, equal to the reduction which the charge on the open end produces in the number of ions which would have entered through the surface represented in section by *JRW* in the absence of the field. Since, on account of the induced charge, no ions enter the shaded region, the integral in question, multiplied by nv, is equal to the total number of ions which would have passed through the surface represented by *JRW* in the absence of the field, and thus represents more than the number which would have passed through the annulus *SW* in the absence of the induced charge, for the point *J* will lie, in general, at a greater distance from the axis of the tube than will the point *S*. We shall find that, when the potential-gradient is small, the shaded region is small, and the reduction in the number of ions entering the cylinder is closely represented by the value of $nv\int\int FdS$ taken over the complete open end of the tube, and while this approximation holds, the reduction in question will be proportional to the potential-gradient. As the potential-gradient, and consequently the shaded region, increases more and more, however, the curve representing the percentage reduction in the ions entering bends more and more towards parallelism with the potential-gradient axis.

In order to render the above explanation of the phenomenon more complete, and for the further interest of the mathematical problem itself, it seems worth while to work out a special case in detail.

The Mathematical Problem.

The complete mathematical problem would involve finding the magnitude and distribution of the charge on the end of the instrument as a function of the potential-gradient (the total charge would of course be proportional to the potential-gradient) and then finding the position of the ring represented by *RR* as a function of the potential-gradient and of the air velocity. The correction to the number of ions entering per second may then be calculated as the sum of two parts: (*a*) a part equal to $nv\int\int FdS$ taken over any cap bounded by the circle *RR*, and (*b*) a part equal to $\pi(c^2 - r^2)nU$, where *U* is the vertical velocity of the air, and *c* and *r* are respectively the radii of the tube and of the ring *RR*.

The problem of calculating the amount and distribution of charge density on the open end of a system of the kind to be found in the Ebert ion-counter would be, in general, too difficult. Since, however, we are merely interested in illustrating the law of the variation of the correction with the potential-gradient, we shall, for simplicity, confine our attention

to a case where the air moves everywhere parallel to the axis with a uniform velocity, and where the induced charge approximates to an infinitely thin ring at the top of the cylinder.

The ring *RR* is determined by the condition that the resultant ionic velocity there is zero for all directions, for the ring *RR* marks the place where the stream lines divide. Hence, at all points such as *R*, we have,

$$-\frac{\partial V}{\partial z}v = U, \qquad \frac{\partial V}{\partial y} = 0,$$

where the axis of z is the axis of the cylinder, and the axis of y is perpendicular thereto. To be perfectly exact, we should regard U as the downward velocity of the air minus the velocity given to the ions by the uniform atmospheric potential-gradient upon which the effects of the induced charge are superposed.

Confining ourselves then to an infinitely thin ring of charge of total

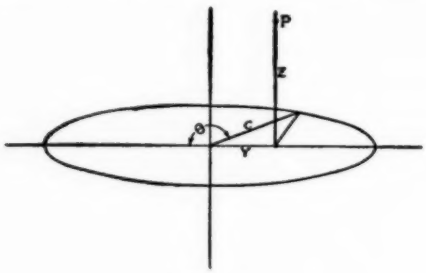

Fig. 2.

amount M, and taking the origin of coördinates at the center of the ring, we have for the potential at any point P

$$V = \frac{M}{2\pi}\int_0^{2\pi} \frac{d\theta}{(z^2 + c^2 + y^2 + 2cy\ \mathrm{Cos}\ \theta)^{1/2}},$$

where c is the radius of the ring, and the angle θ is measured in the plane of the ring.

If we write

$$\varphi = \frac{\theta}{2}, \qquad \frac{y}{c} = x, \qquad \frac{z}{c} = \epsilon \tag{1}$$

and

$$k^2 = \frac{4x}{\epsilon^2 + (x+1)^2} \tag{2}$$

we can readily transform the above value of V to

$$V = \frac{2M}{c\pi}\,\frac{1}{(\epsilon^2 + 1 + x^2 + 2x)^{1/2}}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(1 - k^2\ \mathrm{Sin}^2\ \varphi)^{1/2}}. \tag{3}$$

Our first purpose is to find the locus of the points such as R, *i.e.*, the points for which $\partial V/\partial y$ is zero.

Using relations (1), (2), and (3), and making several reductions, the details of which it is unnecessary to state at length, we find for the locus $\partial V/\partial y = 0$,

$$1 - \frac{k_0^2}{2}(x_0 + 1) = (1 - k_0^2)\frac{K(k_0)}{E(k_0)}, \tag{4}$$

where subscript zero is to be taken as referring to the locus, and $K(k_0)$ and $E(k_0)$ are the elliptic integrals of the first and second kind, viz.,

$$K(k_0) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{(1 - k_0^2 \operatorname{Sin}^2 \varphi)^{1/2}}, \qquad E(k_0) = \int_0^{\frac{\pi}{2}} (1 - k_0^2 \operatorname{Sin}^2 \varphi)^{1/2} d\varphi.$$

In a similar manner we find, by differentiating (3) with respect to z,

$$-\frac{\partial V}{\partial z} = \frac{M}{4\pi c^2}\frac{\epsilon k^3}{x^{3/2}(1 - k^2)}E(k). \tag{5}$$

By assigning different values to k_0, the integrals K and E may be evaluated; the corresponding values of x_0 may be calculated from (4), and the corresponding values of ϵ_0 and consequently of z_0 from equations (2) and (1). To each point on the locus there will of course correspond one value of M which will make $-v\partial V/\partial z = U$ for that point, and the value of M will be determined, from (5), by

$$\frac{Mv}{4\pi c^2 U}\cdot\frac{\epsilon_0 k_0^3}{x_0^{3/2}(1 - k_0^2)}E(k_0) = 1. \tag{6}$$

The dome-shaped dotted line of Fig. 1 represents the actual locus as plotted from equation (4).

If y_0 determines a point on the locus corresponding to a certain value of M, we have, for that value of M,

$$\begin{aligned} \Delta N &= \pi(c^2 - y_0^2)nU + vn\int_0^{y_0} 2\pi y\left(-\frac{\partial V}{\partial z}\right)dy, \\ N &= \pi c^2 nU, \\ \frac{\Delta N}{N} &= (1 - x_0^2) + \frac{2v}{c^2 U}\int_0^{y_0} y\left(-\frac{\partial V}{\partial z}\right)dy, \end{aligned} \tag{7}$$

or, using (5),

$$\frac{\Delta N}{N} = (1 - x_0^2) + \frac{vM}{2\pi c^2 U}\int_0^{x_0}\frac{x\epsilon k^3}{x^{3/2}(1 - k^2)}E(k)dx, \tag{8}$$

or, from (6),

$$\frac{\Delta N}{N} = (1 - x_0^2) + \frac{2x_0^{3/2}(1 - k_0^2)}{\epsilon_0 k_0^{3/2}E(k_0)}\int_0^{x_0}\frac{\epsilon k^3}{x^{1/2}(1 - k^2)}E(k)dx, \tag{9}$$

with the corresponding value of M given by (6).

The values of $\Delta N/N$ for various values of M were calculated in the following manner: Seventeen values of k_0 corresponding to values of $\sin^{-1} k_0$ ranging from 20° to 89° were taken. For each of these the corresponding values of ϵ_0 and x_0 appropriate to the locus defined by (4) were obtained. For each set of values of x_0, ϵ_0 and k_0 the coefficient of the integral in (9) was calculated, and the integral itself, up to the particular value of x_0, was obtained graphically.[1] Thus it was possible to obtain the values of $\Delta N/N$ corresponding to the relation $-v\partial V/\partial z = U$ holding for any value of x_0 on the locus. The corresponding values of $Mv/4\pi c^2 U$

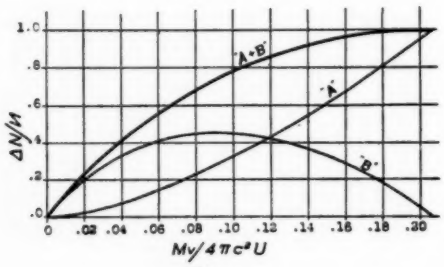

Fig. 3.

were then calculated from (6), so that it was possible to plot $\Delta N/N$ against $Mv/4\pi c^2 U$, which quantity is proportional to the potential gradient, since M is proportional to the potential gradient.[2] The result is shown in Fig. 3, by the line labelled $A + B$. The lines A and B correspond respectively to the first and second terms of equation (9).

It is of interest to observe that, on the basis of the simpler theory referred to at the beginning of this paper, we should have $\Delta N/N$ given by the second member of (8), but with the integral extended from $x = 0$ to $x = 1$. Writing $\Delta N'$ for the error as calculated on this basis we have

$$\frac{\Delta N'}{N} = \left(\frac{vM}{2\pi c^2 U}\right) \int_0^1 \frac{\epsilon k^3}{x^{1/2}(1 - k^2)} E(k)dx,$$

so that, writing

$$\frac{vM}{4\pi c^2 U} = \zeta,$$

[1] For the purpose of calculating the integral for any value of x_0, the values of ϵ, k, and x corresponding to smaller values of x on the locus (4) were, for convenience, used in plotting. This procedure corresponds to taking the integral over the portion of the surface of the locus bounded by the circle $x = x_0$. This integral, since it is proportional to the integral in equation (7), would, of course, have amounted to the same thing if taken over any other surface bounded by the circle $x = x_0$ lying on the locus.

[2] As already remarked, U depends slightly upon the potential-gradient, since, even in the absence of any induced charge on the top of the ion-counter, the potential-gradient would cause the downward velocity of the negative ions to be less than the downward velocity of the air. The velocity due to the potential-gradient is only of the order 2 cm per second, so that the correction on this account is negligible.

we have

$$\frac{d}{d\zeta}\left(\frac{\Delta N'}{N}\right) = 2\int_0^1 \frac{\epsilon k^3}{x^{1/2}(1-k^2)} E(k)dx.$$

The corresponding ζ-derivative for the second member of (8) is

$$2\int_0^1 \frac{\epsilon k^3}{x^{1/2}(1-k^2)} E(k)dx + 2\,Lt_{x_0=1}\,\zeta\,\frac{\epsilon_0 k_0^3}{x_0^{1/2}(1-k_0^2)} E(k_0)\frac{dx_0}{d\zeta}. \quad (10)$$

In view of (6), the second member of (10) is equal to $2Lt_{x_0=1}\,dx_0/d\zeta$. Again, we can readily show, from (4), that k_0, and consequently $E(k_0)$ tend to unity as x_0 tends to unity, so that, in the neighborhood of $x_0 = 1$, we may write $x_0 = 1 + \nu$, $k_0 = 1 + \beta$, $E(k_0) = 1 + \eta$, where β and η vanish with ν.

Utilizing these expressions in (6), after substituting therein for ζ, we can readily show, with the aid of (2) and (4), that

$$Lt_{x_0=1} dx_0/d\zeta = 0.$$

Hence, the second member of (10) is zero in the limit, and the slope of the curve B at the origin is equal to the slope of the straight line representing $\Delta N'/N$ plotted against $vM/4\pi c^2 U$, which straight line is the graph which we should obtain on the basis of the simpler theory formerly discussed.

It is of course too much to expect that a calculation on the basis of the simplified assumption of a thin ring of charge will duplicate the experimental results, and all that is intended in the present paper is a mathematical examination of a special case which, while of simplified type, has in it many of the physical features applicable to the actual case, and which leads, moreover, to the type of departure from linearity found by Mr. Mackell for the relation between $\Delta N/N$ and the potential-gradient.

It is hardly necessary to remark that the more complete theory here sketched is in complete harmony with the earlier theory and with experiment, as indicating an absence of error in measurements of the *positive* ionic density, the potential-gradient being supposed of the normal sign.

Assistance in the numerical computation in this paper has been provided through the research funds of the Executive Committee of the Graduate School of the University of Minnesota. The writer wishes to take this opportunity of expressing his sincere thanks to the committee.

DEPARTMENT OF PHYSICS,
UNIVERSITY OF MINNESOTA,
September 27, 1922.

MAGNETIZATION IN WEAK FIELDS AS A FUNCTION OF FREQUENCY.

By Leigh Page.

Abstract.

Theory of intensity of magnetization for high frequencies of the impressed field.—The attempt is made to obtain a theoretical explanation of the known variation of the permeability of paramagnetic substances with frequency. The magnetic properties of the atom are supposed due to rings of electrons revolving about a nucleus, the field is supposed weak, and damping is neglected. (1) *Paramagnetic solid.* A statistical distribution in direction of the axes of the elementary magnets of which a magnetic solid is composed takes place as the result of collisions between molecules. In the time intervening between successive collisions the elementary magnets follow the impressed field, subject to such restraints as are imposed by the molecular field. From these considerations an expression for the permeability is deduced. The permeability is practically independent of frequency at first, then rises to a maximum for a frequency comparable with the number of collisions per unit time, decreases to a value less than one, and finally converges asymptotically to unity for an infinite frequency. (2) *Paramagnetic gas.* The molecular field is absent, and in consequence there is no maximum in the permeability. After decreasing to a value slightly less than one, the permeability approaches unity asymptotically.

IT is well known that the permeability of paramagnetic substances decreases with increasing frequency of the impressed field, becoming unity for frequencies as great as those of light. The usual explanation is that the inertia of the ultimate magnetic entity is so great that it is unable to follow sufficiently rapid oscillations of an impressed magnetic field. The object of the present paper is to develop a theory which will exhibit a relation between the intensity of magnetization I and the frequency $\omega/2\pi$ of the impressed field for the whole range of frequency. The magnetic properties of the atom will be supposed to be due to rings of electrons revolving about a central positive nucleus, although the results obtained are in no way limited to this particular type of magneton. The treatment will be limited to the case of fields so weak that the intensity of magnetization may be considered proportional to the first power of the field strength throughout an oscillation of the field. In so far as the propagation of light through paramagnetic media is concerned, this limitation is allowable in view of the fact that the maximum field strength in sunlight is only 0.04 gauss.

Consider a region τ whose linear dimensions are small compared to c/ω, where c is the velocity of light, but large enough to contain a great many atoms. Divide this region into as many equal smaller volumes τ_i as there are atoms in τ, each of these smaller volumes containing a single atom. If N denotes the number of atoms per unit volume, τ_i is the reciprocal of N.

On the electron theory of matter, the magnetic induction $\boldsymbol{B}$ in the region τ is the average value of the total magnetic intensity $\boldsymbol{H}$ in this region, that is, in vector notation,

$$\boldsymbol{B} = \frac{1}{\tau}\int_{\tau} \boldsymbol{H} d\tau = \frac{1}{\tau}\sum_{i}\int_{\tau_i} \boldsymbol{H} d\tau. \tag{1}$$

In each of the smaller regions τ_i divide $\boldsymbol{H}$ into three parts, $\boldsymbol{H}_1$, $\boldsymbol{H}_1'$ and $\boldsymbol{H}_2$. $\boldsymbol{H}_2$ is the field due to the atom contained in the region τ_i. $\boldsymbol{H}_1'$ is the field produced by neighboring atoms when the medium is in the unmagnetized state. This field, which, in the case of a solid, will be supposed to be large compared to the other fields involved, exerts a restoring torque on an atom which has had its magnetic axis displaced by the action of an external force. It varies greatly in direction from atom to atom, but its magnitude will be assumed to be approximately the same for all atoms. $\boldsymbol{H}_1$ is the field $\boldsymbol{L}$ due to external sources[1] plus the field due to the magnetization of the medium. Thus the changes in $\boldsymbol{H}_1'$ due to magnetization of the medium are included in $\boldsymbol{H}_1$. For isotropic media such as are considered here $\boldsymbol{H}_1$ has everywhere the direction of $\boldsymbol{L}$ and has a constant magnitude throughout the region τ.

Now equation (1) becomes

$$\boldsymbol{B} = \frac{1}{\tau}\sum_{i}\left\{\int_{\tau_i}\boldsymbol{H}_1 d\tau + \int_{\tau_i}\boldsymbol{H}_1' d\tau + \int_{\tau_i}\boldsymbol{H}_2 d\tau\right\}$$

$$= \boldsymbol{H}_1 + \frac{1}{\tau}\sum_{i}\int_{\tau_i}\boldsymbol{H}_1' d\tau + \frac{1}{\tau}\sum_{i}\int_{\tau_i}\boldsymbol{H}_2 d\tau.$$

It is clear from the definition of $\boldsymbol{H}_1'$ given above that

$$\sum_{i}\int_{\tau_i}\boldsymbol{H}_1' d\tau = 0,$$

and if the small region τ_i is assumed equivalent to a sphere of the same volume, it is easily shown that for a ring of electrons of magnetic moment $\boldsymbol{M}$ contained in it

$$\int_{\tau_i}\boldsymbol{H}_2 d\tau = \frac{2}{3}\boldsymbol{M},$$

or $\frac{1}{\tau}\sum_{i}\int_{\tau_i}\boldsymbol{H}_2 d\tau = \frac{2}{3}N\overline{\boldsymbol{M}} = \frac{2}{3}\boldsymbol{I}$. Therefore $\boldsymbol{H}_1 = \boldsymbol{B} - \frac{2}{3}\boldsymbol{I} = \boldsymbol{L} + \frac{1}{3}\boldsymbol{I}$.

[1] The letter $\boldsymbol{H}$ is used to designate the external field by most writers.

since, in the Heaviside-Lorentz rational units used in this paper, $\boldsymbol{B} = \boldsymbol{L} + \boldsymbol{I}$.

Now consider a single atom. The magnetic field to which it is subject is

$$\boldsymbol{H}_1 + \boldsymbol{H}_1' = \boldsymbol{L} + \frac{1}{3}\boldsymbol{I} + \boldsymbol{H}_1'. \tag{3}$$

A. Paramagnetic Solid.

In a paramagnetic solid $\boldsymbol{H}_1'$ is large compared to the other fields involved, and the magnetic axis of an atom will never deviate far from the direction of this field. Let α (Fig. 1) be the angle which $\boldsymbol{H}_1'$ makes with $\boldsymbol{H}_1$, and let θ be the angle which the projection of the magnetic moment $\boldsymbol{M}$ on the plane of the figure makes with $\boldsymbol{H}_1'$. Then if A is the moment of inertia about a diameter of the ring of electrons which is responsible for the magnetic properties of the atom, and if $H_1 = H_0 \cos \omega t$, the equation of motion of the atom is

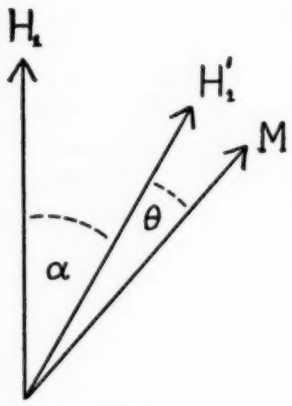

Fig. 1.

$$\frac{d^2\theta}{dt^2} + n^2\theta = -a \cos \omega t, \tag{4}$$

where $n^2 = (MH_1'/A)$, and $a = (MH_0/A) \sin \alpha$.

Solving the equation of motion and determining the constants of integration in terms of the angular displacement θ_0 and angular velocity w_0 at time t_0,

$$\theta = \frac{a}{\omega^2 - n^2}\left\{\cos \omega t - \cos \omega t_0 \cos n(t - t_0) + \frac{\omega}{n}\sin \omega t_0 \sin n(t - t_0)\right\} + \theta_0 \cos n(t - t_0) + \frac{w_0}{n}\sin n(t - t_0).$$

Put $\tau \equiv t - t_0$. Then τ represents the time which has elapsed since the initial conditions θ_0 and w_0 were existent. In terms of τ,

$$\theta = \frac{a}{\omega^2 - n^2}\left\{\cos \omega t - \cos n\tau \cos \omega(t - \tau) + \frac{\omega}{n}\sin n\tau \sin \omega(t - \tau)\right\} + \theta_0 \cos n\tau + \frac{w_0}{n}\sin n\tau. \tag{5}$$

Now consider the number of atoms per unit volume for which $\boldsymbol{H}_1'$ makes an angle between α and $\alpha + d\alpha$ with $\boldsymbol{H}_1$, $\delta N = (1/2)N \sin \alpha d\alpha$. Of these, the number which suffered their last collision at a time between

t_0 and $t_0 + d\tau$ is $\delta^2 N = \delta N e^{-\nu\tau}\nu d\tau = \frac{1}{2} N e^{-\nu\tau}\nu d\tau \sin \alpha d\alpha$, where ν is the average number of collisions per unit time.

As a result of the collision the magnetic axes of these atoms were distributed statistically in accordance with the field existing at the time t_0 at which the collision occurred. This field was $\boldsymbol{H}_1 + \boldsymbol{H}_1' = \boldsymbol{H}_0 \cos \omega(t - \tau) + \boldsymbol{H}_1'$, and the potential energy U due to it of an atom the projection of whose axis made an angle θ_0 with $\boldsymbol{H}_1'$ was

$$U = - H_1 M \cos \alpha + H_1 M \sin \alpha \cdot \theta_0 + \frac{1}{2}(H_1 M \cos \alpha + An^2)\theta_0^2 + \cdots.$$

Put $2a_1a_2 \equiv H_1 M \sin \alpha$, $2a_2^2 \equiv H_1 M \cos \alpha + An^2$.

Then

$$\delta^3 N = Ce^{-[(a_2\theta_0 + a_1)^2 + \frac{1}{2}Aw_0^2]/kT} d\theta_0 dw_0 \tag{6}$$

is the number of atoms the projections of whose axes made angles between θ_0 and $\theta_0 + d\theta_0$ with $\boldsymbol{H}_1'$ and whose angular velocities lay between w_0 and $w_0 + dw_0$ as a result of the collision. To evaluate the constant C, this expression must be integrated with respect to θ_0 and w_0, and the result put equal to $\frac{1}{2} N e^{-\nu\tau}\nu d\tau \sin \alpha d\alpha$.

The limits of integration of w_0 are $-\infty$ and $+\infty$, while θ_0 is to be integrated from $-\epsilon$ to $+\epsilon$, where ϵ is a small angle. As $An^2\epsilon^2/kT$ is presumably large compared with unity, however, the error made if the exponent is allowed to range from $-\infty$ to $+\infty$ is negligible. Hence, integrating both θ_0 and w_0 between the limits $-\infty$ and $+\infty$ and substituting the value of C in (6),

$$\delta^3 N = \frac{Na_2\sqrt{2A}}{4\pi kT} e^{-\nu\tau}\nu d\tau e^{-[(a_2\theta_0 + a_1)^2 + \frac{1}{2}Aw_0^2]/kT} \sin \alpha d\alpha d\theta_0 dw_0. \tag{7}$$

During the time τ which has elapsed since the collision occurred, these atoms have oriented themselves in accordance with the equation of motion (4) already written down, and at the time t their axes make the angle with $\boldsymbol{H}_1'$ given by equation (5). At this time the magnetic moment in the direction of the impressed field due to one of them is $M \cos(\alpha + \theta) = M \cos \alpha - M \sin \alpha \cdot \theta$ as θ is small. Substituting the value of θ from (5), multiplying by (7), and integrating with respect to θ_0, w_0, and α, the portion of the intensity of magnetization I at the time t due to those atoms which suffered their last collision at a time earlier by an amount between τ and $\tau + d\tau$ is

$$\delta I = NMe^{-\nu\tau}\nu d\tau \left[\left\{ \frac{1}{x} + \frac{1}{2}\left(1 - \frac{1}{x^2}\right) \log \frac{1 + x}{1 - x} \right\} \cos n\tau \right.$$
$$\left. - \frac{2}{3}\frac{MH_0}{A(\omega^2 - n^2)} \left\{ \cos \omega t - \cos n\tau \cos \omega(t - \tau) + \frac{\omega}{n} \sin n\tau \sin \omega(t - \tau) \right\} \right],$$

where $x \equiv (MH_1/An^2) = (MH_0/An^2) \cos \omega(t - \tau)$.

As x is small compared to unity, this becomes

$$\delta I = \frac{2}{3}\frac{NM^2H_0}{An^2} e^{-\nu\tau}\nu d\tau[\cos n\tau \cos \omega(t-\tau) - \frac{n^2}{\omega^2 - n^2}\left\{\cos \omega t - \cos n\tau \cos \omega(t-\tau) + \frac{\omega}{n}\sin n\tau \sin \omega(t-\tau)\right\}].$$

Summing up over all τ's from zero to infinity, and writing α for the ratio of ω to ν, and β for the ratio of n to ν,

$$I = \sigma K H_0 \cos(\omega t - \eta), \tag{8}$$

where $\sigma \equiv \frac{2}{3}\frac{NM^2}{An^2}$, $K^2 \equiv \frac{1 + \alpha^2 + 2\beta^2 + \beta^4}{(1 + \alpha^2 - \beta^2)^2 + 4\beta^2}$,

$$\tan \eta \equiv \frac{\alpha(1 + \alpha^2 + \beta^2)}{(1 + \alpha^2 - \beta^2)(1 - \beta^2) + 4\beta^2}.$$

This equation shows that the intensity of magnetization is in general out of phase with the impressed field. The factor K contained in the amplitude has a pronounced maximum if β is large compared to unity, that is, if the natural frequency due to the molecular field is large compared to the number of collisions per unit time. As α becomes large compared to unity and to β, K decreases, approaching zero as the frequency of the impressed field becomes indefinitely great.

The frequency of the field H_1 must be the same as that of the impressed field L, but these two fields may not be in phase. Therefore put $L = L_0 \cos(\omega t + \delta) = L_0 \cos \omega t'$. Then, substituting the value of I given by (8) in (2), $H_0 \cos \omega t = L_0 \cos(\omega t + \delta) + \frac{1}{3}\sigma K H_0 \cos(\omega t - \eta)$, whence

$$I = \epsilon L_0 \cos(\omega t' - p); \tag{9}$$

where the susceptibility ϵ is $\sigma K(1 - \frac{2}{3}\sigma K \cos \eta + \frac{1}{9}\sigma^2 K^2)^{-1/2}$, and the phase angle p is given by $\tan p = \tan \eta/(1 - \frac{1}{3}\sigma K \sec \eta)$. Furthermore, $B = L + I$

$$= L_0\left\{\cos \omega t' + \frac{\sigma K}{\sqrt{1 - \frac{2}{3}\sigma K \cos \eta + \frac{1}{9}\sigma^2 K^2}} \cos(\omega t' - p)\right\}, \tag{10}$$

$$= \mu L_0 \cos(\omega t' - q);$$

where the permeability μ and phase angle q are given by

$$\mu = \sqrt{\frac{1 + \frac{4}{3}\sigma K \cos \eta + \frac{4}{9}\sigma^2 K^2}{1 - \frac{2}{3}\sigma K \cos \eta + \frac{1}{9}\sigma^2 K^2}},$$

and

$$\tan q = \frac{\sigma K \sin \eta}{1 + \frac{1}{3}\sigma K \cos \eta - \frac{2}{9}\sigma^2 K^2}.$$

For a ferro-magnetic substance σ must be only slightly less than 3. Putting $3(1 - \chi)$ for σ, y for $(1 + \alpha^2)$, and substituting the expressions (8) for K and η into the expressions for μ and $\tan q$, it is found that

$$\mu^2 = \frac{y^2 - 2y[3\beta^2 - 4 - \chi(2\beta^2 - 6) - 2\chi^2] + 9\beta^4 + 24\beta^2 - 4\chi(3\beta^4 + 7\beta^2) + 4\chi^2(\beta^4 + 2\beta^2)}{y^2 - y[1 + 2\chi\beta^2 - \chi^2] + 2\chi\beta^2 + \chi^2(\beta^4 + 2\beta^2)},$$

$$\tan q = \frac{3\alpha\{y[1 - \chi] + \beta^2(1 - \chi)\}}{y^2 - y[3\beta^2 + 1 - \chi(\beta^2 + 3) + 2\chi^2] + 3\beta^2 + \chi(3\beta^4 + 5\beta^2) - 2\chi^2(\beta^4 + 2\beta^2)}.$$

The table is based on a value of 10 for β and 0.03 for χ. It is seen that the permeability remains practically constant until the frequency of the impressed field becomes equal to a tenth of the number of collisions per unit time. Then the permeability rises to a maximum equal in amount to about one and three quarters its original value, after which it decreases to a value less than unity, and finally approaches unity asymptotically as the frequency becomes indefinitely great. A maximum such as that indicated by the theory has been observed experimentally by Wwedensky and Theodortschik.[1]

TABLE I.

Paramagnetic Solid.

α.	$(1 + \alpha^2)$.	K.	$\cos \eta$.	$\sigma = 3(1 - .03)$.	
				μ.	$\tan q$.
0	1	1.00	1.00	98.0	0
0.03162	1.001	1.00	1.00	98.0	0.01
0.1000	1.01	1.00	1.00	98.3	0.03
0.3162	1.1	1.00	1.00	100.7	0.11
1.000	2	1.01	1.00	130	0.49
1.414	3	1.02	1.00	165	1.36
1.581	3.5	1.02	1.00	172	2.9
1.732	4	1.03	1.00	165	25
2.000	5	1.04	1.00	129	− 2.3
3.000	10	1.10	1.00	43	− 0.57
7.000	50	1.88	0.98	5.4	− 0.03
9.950	100	5.07	0.20	2.1	− 0.31
17.29	300	0.51	−0.94	0.23	11.2
22.34	500	0.26	−0.94	0.45	0.38
31.61	1,000	0.12	−0.93	0.72	0.14
100.0	10,000	0.01	−0.71	0.97	0.03
316.2	100,000	0.00	−0.31	1.00	0.01

[1] B. Wwedensky and K. Theodortschik, Ann. d. Phys., 68, 463 (1922).

The phase angle q increases with frequency, passing into the second quadrant at a frequency slightly greater than that required for maximum permeability. After approaching π, q recedes, returning to zero at indefinitely great frequencies.

The values given in the table must not be considered as more than representative of the relation existing between the magnetic properties under consideration and frequency; for the constants σ and β are not known and may have values different from those assumed in computing the numbers given in the table. Moreover, the numerical coefficient of I in (3) can be considered as only very roughly equal to one third. Finally, the theory itself is incomplete in that damping of the elementary magnets has not been taken into account. Nevertheless, the general sequence of values in the table seems to be in accord with experimental observation, and whatever the values of σ and β may be, it is clear that μ converges to unity as the frequency is indefinitely increased.

B. Paramagnetic Gas.

In a gas the field $\boldsymbol{H}_1'$ is presumably negligible, and hence there are no quasi-elastic torques tending to hold the axes of the elementary magnets in fixed directions. Therefore the angular velocities due to thermal agitation may cause the atoms to rotate through large angles between successive collisions. The fact that the angular velocities are distributed statistically, however, makes it seem probable that the resultant effect is sensibly the same as if the elementary magnets had no other angular velocities than those acquired as a result of the action of the impressed field. In the following development this will be assumed to be the case.

If α (Fig. 1) is the angle which the magnetic axis of an atom makes with the direction of the impressed field at the time t_0, the equation of motion of the atom under the influence of the simple harmonic field $H_1 = H_0 \cos \omega t$ is

$$\frac{d^2\theta}{dt^2} = -a \cos \omega t, \tag{11}$$

where a has the same meaning as in Eq. (4). Hence

$$\theta = \frac{a}{\omega^2}\{\cos \omega t - \cos \omega(t - \tau) + \omega\tau \sin \omega(t - \tau)\}, \tag{12}$$

where $\tau \equiv t - t_0$, and the constants of integration have been chosen so as to make θ and the angular velocity zero at time t_0.

As before, if N is the number of atoms per unit volume, the number of these which suffered a collision at a time between t_0 and $t_0 + d\tau$ is

$$\delta N = Ne^{-\nu\tau}\nu d\tau.$$

As a result of the collision the magnetic axes of these atoms were distributed statistically in accordance with the field $H_0 \cos \omega(t - \tau)$, existing at the time t_0 at which the collision occurred. Hence, putting $x \equiv (MH_1/kT) = (MH_0/kT) \cos \omega(t - \tau)$, the number of these atoms whose axes made angles between α and $\alpha + d\alpha$ as a result of the collision is readily found to be

$$\delta^2 N = \frac{1}{2} \delta N \frac{x}{\sinh x} e^{x \cos \alpha} \sin \alpha d\alpha. \tag{13}$$

During the time τ which has elapsed since the collision occurred, these atoms have turned through the angle θ, and the magnetic moment in the direction of the impressed field due to each of them is

$$M \cos (\alpha + \theta) = M \cos \alpha - M \sin \alpha \cdot \theta.$$

Putting in the value of θ given by (12), multiplying by (13), and integrating with respect to α, the portion of the intensity of magnetization I at the time t due to those atoms which suffered their last collision at a time earlier by an amount between τ and $\tau + d\tau$ is

$$dI = NMe^{-\nu\tau} \nu d\tau \left[\frac{\cosh x}{\sinh x} - \frac{1}{x} \right] \times \left[1 - \frac{2}{x} \frac{MH_0}{A\omega^2} \{\cos \omega t - \cos \omega(t - \tau) + \omega\tau \sin \omega(t - \tau)\} \right].$$

As x is small compared to unity, this becomes

$$dI = \frac{2}{3} \frac{NM^2 H_0}{A\omega^2} e^{-\nu\tau} \nu d\tau \times \left[\frac{A\omega^2}{2kT} \cos \omega(t - \tau) - \cos \omega t + \cos \omega(t - \tau) - \omega\tau \sin \omega(t - \tau) \right].$$

Summing up over all τ's, and writing α for the ratio of ω to ν and r for the ratio of $A\nu^2$ to $2kT$,

$$I = \sigma K H_0 \cos (\omega t - \eta), \tag{14}$$

where $\sigma \equiv (2/3)(NM^2/A\nu^2)$, $K \equiv \sqrt{1 + 2r + r^2(1 + \alpha^2)}/(1 + \alpha^2)$, $\tan \eta \equiv [2\alpha + r\alpha(1 + \alpha^2)]/[1 - \alpha^2 + r(1 + \alpha^2)]$.

For a gas, r is probably of the order of 10^{-10}, and hence, as a sufficiently close approximation,

$$K = \frac{1}{1 + \alpha^2}, \qquad \tan \eta = \frac{2\alpha}{1 - \alpha^2}.$$

TABLE II.

Paramagnetic Gas.

α.	$1 + \alpha^2$.	K.	$\cos \eta$.	$\sigma = 1.5$.	
				μ.	tan q.
0	1	1.00	1.00	4.00	0
.1000	1.01	.99	.98	3.78	.30
1.000	2	.50	.00	1.08	.86
3.000	10	.10	− .90	.89	.09
9.950	100	.01	− .99	.98	.003
31.61	1,000	.00	−1.00	1.00	0

The table, which is based on the admittedly high value of 1.5 for σ in order to make the course of the curve more evident, shows that μ decreases with increasing frequency, reaching a value a little under unity, and then increases asymptotically to one. The values of tan q increase to a maximum, and then decrease to zero, but q does not leave the first quadrant.

SLOANE PHYSICS LABORATORY,
YALE UNIVERSITY,
December 26, 1922.

LINES OF ELECTRIC FORCE OF A MOVING CHARGE ON THE EMISSION THEORY.

By Felix E. Hackett.

Abstract.

Simplified method of determining the lines of force of a moving electric charge, on the emission theory of Page and Bateman.—This theory supposes that the charge emits streams of elements which move with the velocity of light, and constitute the lines of force. It leads to a vector equation for the directions of emission. By a transformation a simpler vector equation is derived from which the solutions previously given for variable rectilinear and uniform circular motion are more easily obtained. These solutions are both shown to conform, each in a special way, to the Einstein law of composition of velocities for uniform motion.

LEIGH PAGE [1] has shown how the equations of the electromagnetic field may be derived on the assumption that an electric charge is a source continuously emitting non-homogeneities which may be termed moving elements. "The nature of these moving elements is immaterial for the purpose of the theory other than each must be susceptible of continuous identification and must move in a straight line with the velocity of light. *A line of electric force* is defined as the locus of a stream of moving elements emerging from a single source." [2]

The same representation has been used and developed by Bateman.[3, 4] By this means he obtained the equations for the lines of electric force for a moving electric charge from the expression for the electric field derived from the Lorentz retarded potentials.

Let l, m, n be the direction-cosines of emission of the *moving elements*, or, as Bateman has termed them, *light particles* for any given line of force at any time τ where ξ, η, ζ are the coördinates of the electric charge at any time τ, then the equations of the line of force at any subsequent time t are given by

$$x = \xi + c(t - \tau)l; \qquad y = \eta + c(t - \tau)m; \qquad z = \zeta + c(t - \tau)n. \qquad (1)$$

The vector $d\mathfrak{p}$ joining the positions of two elements emitted at τ and $\tau + d\tau$ gives the direction of the electric field at any subsequent time t

[1] Amer. Jour. Sci., 38, p. 169, 1914.
[2] Page, Phys. Rev., N.S., Vol. XVIII., p. 292, 1921.
[3] Bull. Amer. Math. Soc., March, 1915.
[4] Phil. Mag., 41, p. 107, 1921.

at the point (x, y, z) which may be denoted by the vector $\boldsymbol{r}$ $(x - \xi, y - \eta, z - \zeta)$ where $r = c(t - \tau)$.

Using the vector $\boldsymbol{c}$ to denote the velocity of the moving element emitted at the time τ and $\boldsymbol{v}$ the velocity of the charge, we have

$$\frac{d\boldsymbol{p}}{d\tau} = \boldsymbol{c} - \boldsymbol{v} - \frac{r}{c}\frac{d\boldsymbol{c}}{d\tau}. \tag{2}$$

Writing $\boldsymbol{f}$ for the vector acceleration of the electric charge e and $\boldsymbol{E}$ for the electric field, the expression for $\boldsymbol{E}$ may be written [1]

$$\boldsymbol{E} = \frac{e(1 - \beta^2)}{4\pi\gamma^2 c\left(1 - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}\right)^3}\left[\boldsymbol{c} - \boldsymbol{v} + \frac{r}{c}\frac{(\boldsymbol{f} \times (\boldsymbol{c} - \boldsymbol{v})) \times \boldsymbol{c}}{c^2(1 - \beta^2)}\right]. \tag{3}$$

Since $d\boldsymbol{p}/d\tau$ is parallel to $\boldsymbol{E}$ we have

$$\frac{d\boldsymbol{c}}{d\tau} = -\frac{[\boldsymbol{f} \times (\boldsymbol{c} - \boldsymbol{v})] \times \boldsymbol{c}}{c^2(1 - \beta^2)}. \tag{4}$$

Bateman has given a complex transformation by which the vector equation (4) is replaced by a Riccatian equation. This equation has been discussed by Murnaghan [2] who has given solutions for some simple forms of motion of the charge.

It is shown in this paper that equation (4) can be thrown into a vector equation of simple form from which solutions can readily be obtained for all the simple types of motion treated by Murnaghan. The solutions are shown to be in accordance with a general assumption made by Page regarding the acceleration of a moving charge and the direction of emission of the moving elements. This assumption has been used by him to deduce equation (4) and so expression (3) for the electric force. The results in this paper also support the application made by Page of the Einstein law for the composition of velocities to accelerated motion, an application which would seem to require justification.

Vector Solution.

Writing

$$\boldsymbol{s} = (\boldsymbol{c} - \boldsymbol{v})\frac{\sqrt{1 - \beta^2}}{1 - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}} \tag{5}$$

we find that (4) becomes

$$\frac{\dot{\boldsymbol{s}}}{\boldsymbol{s}\cdot\boldsymbol{v}} = -\frac{\dot{\boldsymbol{v}}}{c^2 - v^2}. \tag{6}$$

[1] Page, Amer. Jour. Sci., 38, p. 169, 1914.

[2] Amer. Jour. of Math., Vol. 39, April, 1917.

This is a linear vector equation of the form

$$\dot{\boldsymbol{s}} + \Phi \boldsymbol{s} = 0,$$

and admits of a symbolical solution in the form

$$\boldsymbol{s} = [1 - \int \Phi d\tau + \int \Phi d\tau \int \Phi d\tau - \int \Phi d\tau \int \Phi d\tau \int \Phi d\tau \cdots]\boldsymbol{q},$$

where $\boldsymbol{q}$ is any arbitrary vector.

Special cases of motion can be solved by aid of the following relations, which are readily established from (5) by writing $\boldsymbol{c} - \boldsymbol{v} = \boldsymbol{u}$:

$$\boldsymbol{s} \cdot \boldsymbol{v} = \sqrt{c^2 - v^2}\ \sqrt{c^2 - s^2}, \tag{7}$$

$$\boldsymbol{c} - \boldsymbol{v} = \boldsymbol{s}\frac{1 - \dfrac{\boldsymbol{v}\cdot\boldsymbol{c}}{c^2}}{\sqrt{1 - \beta^2}} = \boldsymbol{s}\frac{c^2 - v^2}{c\sqrt{c^2 - v^2} + \boldsymbol{s}\cdot\boldsymbol{v}}. \tag{8}$$

It may be noted that (7) enables us to transform (6) into the symmetrical form

$$\frac{\dot{\boldsymbol{s}}}{\sqrt{c^2 - s^2}} = -\frac{\dot{\boldsymbol{v}}}{\sqrt{c^2 - v^2}}, \tag{9}$$

exhibiting the close relationship which exists between the form of the lines of force and the path of the electric charge.

Rectilinear Motion.

Let the charge move with uniform acceleration along the axis of x. Then if s_x, s_y, s_z denote components along the axes and m', n' are constants, we can write from (6)

$$s_y = cm', \qquad s_z = cn';$$

and from (7)

$$v^2 s_x^2 = (c^2 - v^2)(c^2 - c^2 m'^2 - c^2 n'^2 - s_x^2),$$
$$s_x^2 = (c^2 - v^2)(1 - m'^2 - n'^2) = (c^2 - v^2)l'^2,$$

where $l'^2 + m'^2 + n'^2 = 1$.

$$\therefore \quad \boldsymbol{s}\cdot\boldsymbol{v} = l'v\sqrt{c^2 - v^2},$$

We get then by resolving (8) along the axes and writing c_x, c_y, c_z for components of velocity of emission of the moving elements

$$c_x = cl = \frac{l'(c^2 - v^2)\sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2} + l'v\sqrt{c^2 - v^2}} + v$$
$$= \frac{cl' + v}{1 + \dfrac{vl'}{c}}, \tag{10}$$

$$c_y = cm = \frac{cm'\sqrt{1-\beta^2}}{1+\frac{vl'}{c}}.$$

$$c_z = cn = \frac{cn'\sqrt{1-\beta^2}}{1+\frac{vl'}{c}}.$$

These equations are the formulæ of transformation of velocities c_x, c_y, c_z to an observer in a system K' moving with the charge with velocity v with respect to the system K to which all our equations have hitherto referred. The moving elements are emitted, therefore, with velocities cl', cm', cn' to an observer in K' and, as l', m', n' are constants of integration, the directions of emission are therefore constant to an observer moving with the charge.

If K' is the system when the charged particle has the velocity $\boldsymbol{v}$ and K'' when it has the velocity $\boldsymbol{v} + d\boldsymbol{v}$ or $d\boldsymbol{v}'$ relative to K', Page[1] assumes that "the angle between any particular tube of strain in the immediate vicinity of the charged particle and the direction of $d\boldsymbol{v}$ must have the same value to an observer in K'' when the charge is in K'' as this angle has to an observer in K' when the charge is in K'."

When the acceleration is constant in direction, this assumption requires a direction of emission which appears constant to an observer moving with the charge as shown above.

The values for l, m, n given by Murnaghan appear in different form. We obtain his values if we write, using α and β as arbitrary constants,

$$l'/(\alpha^2+\beta^2-1) = m'/2\beta = n'/2\alpha = (\alpha^2+\beta^2+1)^{-1},$$

giving

$$Rl = (c+v)(\alpha^2+\beta^2) - (c-v),\ Rm = 2\beta(c^2-v^2)^{1/2},\ Rn = 2\alpha(c^2-v^2)^{1/2},$$

where

$$R = (c+v)(\alpha^2+\beta^2) + (c-v).$$

Motion in a Circle with Uniform Speed.

Let a be the radius of a circle described by a charge e in the plane of (xy) and p be its angular velocity. Using $\boldsymbol{X}_1$, $\boldsymbol{Y}_1$, $\boldsymbol{Z}_1$ to represent unit vectors along the axes and referring to the center of the circle as origin and to axes rotating with the electron, we have $\boldsymbol{v} = ap\boldsymbol{Y}_1$, $\dot{\boldsymbol{v}} = -ap^2\boldsymbol{X}_1$, $\dot{\boldsymbol{s}}_f = \dot{\boldsymbol{s}}_m + p(\boldsymbol{Z}_1 \times \boldsymbol{s})$ in accordance with the usual formula for the rate of change of a vector quantity with respect to moving axes,[1] where the notation $\dot{\boldsymbol{s}}_m$ refers to the moving system.

[1] Amer. Jour. Sci., 38, p. 169, 1914.

[2] Whittaker, Analytical Dynamics.

Equation (6) becomes

$$\frac{\dot{\boldsymbol{s}}_m + p(\boldsymbol{Z}_1 \times \boldsymbol{s})}{ap \cdot (\boldsymbol{s} \cdot \boldsymbol{Y}_1)} = \frac{ap^2 \boldsymbol{X}_1}{c^2 - p^2 a^2}.$$

We derive the scalar equations

$$\dot{x} - py = a^2 p^3 y/(c^2 - p^2 a^2), \qquad \dot{y} + px = 0, \qquad \dot{z} = 0.$$

Solving, we find

$$x = A \sin(q\tau - \epsilon), \qquad y = A\omega \cos(q\tau - \epsilon), \qquad z = B, \tag{11}$$

where

$$q = p/\omega, \qquad \omega^2 = 1 - a^2 p^2/c^2 = 1 - v^2/c^2.$$

Using

$$(\boldsymbol{s} \cdot \boldsymbol{v})^2 = (c^2 - s^2)(c^2 - v^2),$$

we get

$$y^2 v^2 = (c^2 - v^2)(c^2 - x^2 - y^2 - z^2),$$

giving

$$B = \sqrt{c^2 - A^2}.$$

Referring to fixed axes X, Y, Z, we get for the components (X, Y, Z) of $\boldsymbol{s}$

$$X = x \cos p\tau - y \sin p\tau, \quad Y = x \sin p\tau + y \cos p\tau, \quad Z = \sqrt{c^2 - A^2},$$
$$\boldsymbol{s} \cdot \boldsymbol{v} = A\omega ap \cos(q\tau - \epsilon). \tag{12}$$

We have from (8)

$$\boldsymbol{c} = \boldsymbol{s}/K + \boldsymbol{v},$$

where

$$K = \frac{c}{\sqrt{c^2 - v^2}} + \frac{\omega A ap \cos(q\tau - \epsilon)}{c^2 - v^2}$$
$$= \frac{1}{\omega} + \frac{\omega A ap \cos(q\tau - \epsilon)}{c^2 \omega^2}. \tag{13}$$

Hence

$$cl = X/K - ap \sin p\tau; \quad cm = Y/K + ap \cos p\tau; \quad cn = \sqrt{c^2 - A^2}/K. \tag{14}$$

This solution can be thrown into the form given by Murnaghan [1] by the substitutions, using his notation

$$g = (1 - \omega)/2\omega, \qquad \tan \epsilon = \beta/\alpha,$$
$$S = \frac{\alpha^2 + \beta^2}{g\omega} + \frac{g}{\omega} - \frac{2ap}{c\omega}(\alpha \cos q\tau + \beta \sin q\tau),$$
$$K = \frac{gS}{\alpha^2 + \beta^2 + g^2},$$
$$A = -\frac{2cg\sqrt{\alpha^2 + \beta^2}}{\alpha^2 + \beta^2 + g^2},$$

[1] Loc. cit.

giving

$$\frac{\sqrt{c^2 - A^2}}{cK} = \frac{g^2 - (\alpha^2 + \beta^2)}{gS} = \frac{1}{S}\left[g - \frac{\alpha^2 + \beta^2}{g}\right] = \frac{T}{S}.$$

Lines of Force Due to Uniform Circular Motion.

Varying the notation used in equation (1) we can write for the equations of the lines of force at the time $t = 0$,

$$\xi = a\cos p\tau - cl\tau; \qquad \eta = a\sin p\tau - cm\tau; \qquad \zeta = -cn\tau.$$

Substituting for l, m, we get from (13), (14)

$$\xi = a\cos p\tau - \tau\left[\frac{A\cos p\tau\sin(q\tau-\epsilon) - A\omega\sin p\tau\cos(q\tau-\epsilon)}{1/\omega + Aap\cos(q\tau-\epsilon)/c^2\omega} - ap\sin p\tau\right],$$

$$\eta = a\sin p\tau - \tau\left[\frac{A\sin p\tau\sin(q\tau-\epsilon) + A\omega\cos p\tau\cos(q\tau-\epsilon)}{1/\omega + Aap\cos(q\tau-\epsilon)/c^2\omega} + ap\cos p\tau\right].$$

For purposes of calculation, it is convenient, since only negative values of τ are permissible, to change the sign of τ and write $\tau = -t$. It will also be found convenient to write $\epsilon = \delta - \pi/2$ and to consider only clockwise rotation. In calculating, we deal only with the past history of the charge and, as t increases, we describe the path anticlockwise; we shall therefore use negative values for p and q. In order to avoid cumbering the notation, we shall use the same symbols and merely change their signs, substituting $-p$ for p and $-q$ for q. We get, then,

$$\xi = a\cos pt + t\left[\frac{A\cos pt\cos(qt-\delta) + A\omega\sin pt\sin(qt-\delta)}{1/\omega + Aap\sin(qt-\delta)/c^2\omega} + ap\sin pt\right],$$

$$\eta = a\sin pt + t\left[\frac{A\sin pt\cos(qt-\delta) - A\omega\cos pt\sin(qt-\delta)}{1/\omega + Aap\sin(qt-\delta)/c^2\omega} - ap\cos pt\right].$$

Writing

$$qt = pt/\omega = \theta/\omega = \varphi + \delta,$$
$$\xi p/c = \xi', \qquad \eta p/c = \eta',$$
$$\omega^2 = 1 - a^2p^2/c^2 = 1 - \beta^2,$$

$$c\sqrt{1-n^2}\cos\alpha = \frac{A\omega\cos(qt-\delta)}{1 + Aap\cdot\sin(qt-\delta)/c^2} = \frac{A\sqrt{1-\beta^2}\cos\varphi}{1 + \beta A\sin\varphi/c}, \qquad (15)$$

$$c\sqrt{1-n^2}\sin\alpha = \frac{A\omega^2\sin(qt-\delta)}{1 + Aap\cdot\sin(qt-\delta)/c^2} + ap = \frac{A\sin\varphi + ap}{1 + \beta A\sin\varphi/c},$$

we get finally

$$\xi' = \beta\cos\theta + \theta\sqrt{1-n^2}(\cos\theta\cos\alpha + \sin\theta\sin\alpha),$$
$$\eta' = \beta\sin\theta + \theta\sqrt{1-n^2}(\sin\theta\cos\alpha - \cos\theta\sin\alpha).$$

It will be observed that this alteration in notation is equivalent to taking as the unit of time—the time required to describe one radian—

$1/p$; the distance traversed by light in this time is the new unit of distance c/p so that the velocity of light is unity. The radius of the circle described has the same numerical value (β) as the velocity of the charge.

When β is small $\alpha = \theta - \delta$ and the equations of the lines of force in the plane of the circle for which $n = 0$ may be written in the simple form

$$\xi' = \beta \cos \theta + \theta \cos \delta,$$
$$\eta' = \beta \sin \theta + \theta \sin \delta.$$

The direction of emission is constant and the charge may be said to carry with it its lines of force which may be regarded as straight.

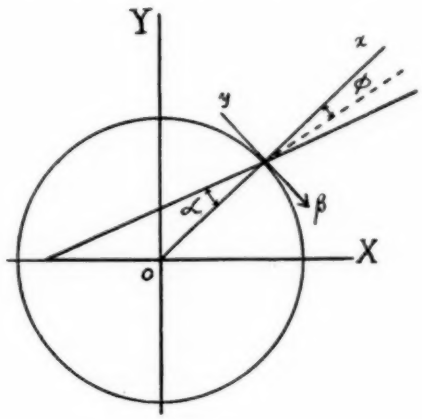

Fig. 1.

In general we have for lines of force in the plane of the circle for which $n = 0$, $A = c$ referred to fixed axes X and Y,

$$\xi' = \beta \cos \theta + \theta \cos (\theta - \alpha),$$
$$\eta' = \beta \sin \theta + \theta \sin (\theta - \alpha),$$

where θ is measured in the usual positive direction,

$\theta - \alpha$ = angle of emission at P as viewed by an observer (K) at the center of the circle,

α = angle of emission with the axis of x where x and y are a set of fixed axes of the system (K) instantaneously coinciding with moving axes at P belonging to a system K' whose direction of motion is $-\beta$ along the y axis.

If we refer the velocity of emission ($c \cos \alpha$, $-c \sin \alpha$, 0) to the system K' by means of the usual relativity formulæ and use (15) in which

we put $A = c$, and $n = 0$, we have

$$c \cos \alpha = v_x = \frac{v_x' \sqrt{1 - \beta^2}}{1 - \beta v_y'/c} = \frac{c \cos \varphi \sqrt{1 - \beta^2}}{1 + \beta \sin \varphi},$$

$$- c \sin \alpha = v_y = \frac{v_y' - c\beta}{1 - \beta v_y'/c} = \frac{- c \sin \varphi - c\beta}{1 + \beta \sin \varphi},$$

from which we may deduce, subject to the validity of the use of these transformations for circular motion, that

$$v_x' = c \cos \varphi, \qquad v_y' = - c \sin \varphi.$$

φ is therefore the angle of emission of the "moving element" of a line of force in the plane of (xy) to an observer travelling with the charge. Since $\varphi = pt/\sqrt{1 - \beta^2} - \delta$, the emission-direction of a moving element will rotate with respect to the radius vector around the axis of z with an angular velocity $p/\sqrt{1 - \beta^2}$ in K units. In consequence the emission-direction for a given line of force may never repeat itself for a given position of the orbit.

In the more general case, using the formulæ of (15) we find that φ is the azimuth of the emission in the plane of (xy) with respect to the moving observer. If we use K' units of time, $\varphi = pt' - \delta$, so that the angular velocity of the emission-directions around the axis of z is p. It is possible therefore for the moving observer to regard the emission-directions as fixed, and to consider that the center of his orbit is rotating with respect to them with angular velocity p.

It can be readily shown from the above results that the condition laid down by Page quoted under rectilinear motion is also satisfied for circular motion.

COLLEGE OF SCIENCE FOR IRELAND,
DUBLIN,
August 10, 1922.

PROCEEDINGS

OF THE

AMERICAN PHYSICAL SOCIETY.

MINUTES OF THE NEW YORK MEETING, FEBRUARY 24, 1923

THE 119th meeting of The American Physical Society was held in Schermerhorn Hall, Columbia University, New York City, on Saturday, February 24, 1923. The president of the Society, Charles E. Mendenhall, presided. The program was completed in the morning session with an attendance of about two hundred.

The regular meeting of the Council was held on the afternoon of Saturday, February 24, 1923, at Columbia University. The following elections were made: *Elected to Fellowship*, James Percy Ault and David A. Keys; *transferred from Membership to Fellowship*, Edgar C. Bain, C. C. Bidwell, G. Faccioli, George E. Gibson and Oliver Holmes Gish; *elected to Membership*, Charles E. Depperman, S.J., Samuel Frantz, Joseph D. R. Freed, W. F. Giauque, Clarence E. Irion, William Sanford Perry, David Rines, Frank Lewis Smith, F. W. Stevens, Richard M. Sutton, Benjamin Toubes, Alva C. Turner, and W. F. Winter.

Abstracts of the twenty-four papers which were presented are given below. Nos. 1, 2, 4, 14, and 24 were read by title.

HAROLD W. WEBB,
Secretary

ABSTRACTS OF PAPERS

1. **Logarithmic and semi-logarithmic coördinator.** R. A. CASTLEMAN, JR., East Falls Church, Va.—A device that combines speed, accuracy, and simplicity for logarithmic and semi-logarithmic plotting is obviously desirable as a laboratory adjunct. Such an apparatus can be simply constructed as follows: Along an edge of a drawing board are fastened two scales, one a log scale, the other a uniform scale; similar scales run along the opposite edges of a transparent edged T-square. Plotting of the above sort may then be made with any desired combination of scales. The slope of any linear relation obtained is measured by means of the uniform scales. Scales can be simply and neatly made by photographing, with a good camera, those on a slide rule. Since power and exponential finding are inverse operations, respectively, to log and semi-log plotting, the extension of the apparatus to include these phases of the problem is obvious. All that is needed is a number of lines of various slopes emanating from a point in the center of the board. If the ones of the log scales and the zeros of the uniform scales fall at this point, the apparatus is direct-reading for all values of the exponent and of the independent variable. Since the divisions on a log scale are much more uniform

than those on a log log scale, the present apparatus has obvious advantages over the log log rule. The apparatus is particularly adapted for quick graphical solution of such common problems as $y = ax^n$ or $y = ae^{nx}$.

2. Free and forced heat convection in gases and liquids. CHESTER W. RICE, General Electric Research Laboratory.—Langmuir's film theory combined with dimensional reasoning gives accurate equations for convection in fluids. For free convection the film thickness is $B = KD(\nu^2/gD^3)^{m/3}$, where K and m are experimental constants, ν the kinematic viscosity of ambient fluid ($= \mu/\rho$), g acceleration of gravity, D a linear dimension characteristic of the hot body. The free convection from a cylinder of length L and diameter D is $W_c = 2\pi L\Delta\varphi/\log_e(2B + D/D)$ watts, where $\Delta\varphi = \int_{T_1}^{T_2} k dT$, k is the heat conductivity of the fluid, and T_2 and T_1 the temperature of the hot body and ambient fluid respectively. Experiments give $m = \frac{1}{3}$ and $K = 3.4$. Free convection tests constitute a method of obtaining the conductivity of liquids. The values written in the form $k = k_0(1 + \alpha t)$ watt cm^{-1} $^\circ C^{-1}$ are: No. 12 Transil oil $k_0 = .0016$, $\alpha = .0095$; Toluene $k_0 = .0011$, $\alpha = .0095$; Glycerine $k_0 = .0027$, $\alpha = .016$; Olive Oil $k_0 = .0017$, $\alpha = .016$, Aniline $k_0 = .0017$, $\alpha = .014$; CCl_4 $k_0 = .0009$, $\alpha = .0095$.

In forced convection $B = KD(\nu D^{-1}v^{-1})^m$, where v is the velocity of fluid. Available experiments are satisfied by taking ν for the geometric mean of surface and ambient temperatures. For cylinders $m = \frac{1}{3}$ and $K = 2.15$ when $\nu D^{-1}v^{-1} < .013$ and $BD^{-1} < .24$ and $m = 2/3$, $K = 4.2$ above these values. For pipes $W_c = 2\pi L\Delta\varphi/\log_e(D/D - 2B)$ watts with $m = 3/4$, $K = 30$ as approximate values. For large surfaces the film thickness will usually be small compared with the body size. For moderate temperature differences we may therefore write $W_c = KA(k/\nu)v\Delta t_{avg.}$ watts, where $K = .003$ approximately. It follows that the forced convection from a plane surface in a mixture of a light with a heavy gas may be considerably greater than for either gas alone. For example, 60 per cent H_2 by volume in O_2 gives approximately 1.8 times the forced convection of either gas alone.

3. A neglected method of physico-chemical research. ARTHUR W. GRAY, Caulk Dental Research Institute, Milford, Delaware.—Volume change accompanying solution, chemical reaction, and crystallization may be conveniently analyzed by division into three consecutive changes which accompany (1) the mutual solution of the reacting substances preceding their combination or decomposition; (2) the atomic rearrangement constituting combination or decomposition; (3) the molecular rearrangement constituting crystallization of the resulting substance. *Dilatometric investigations* with the aid of reaction expansion curves are capable of yielding information concerning diffusion, solution; chemical reaction, and crystallization as they proceed in a substance while it is essentially solid, and concerning the effects of conditions that influence such phenomena. Such investigations might throw light upon the causes of the gradual changes that occur in materials exhibiting ageing phenomena, such as cements, and alloys like duralumin, invar, and various steels after thermal or mechanical treatment. A detailed discussion of general principles of such investigations and their application to the elucidation of hitherto unexplained phenomena will be found in a paper on volume changes in amalgams which will be presented by the author at the meeting of the British Institute of Metals in March, 1923.

4. The distribution of intensity in the broadened Balmer lines of hydrogen. E. O. HULBURT, University of Iowa.—The intensity distribution across the broadened lines $H\beta$, $H\gamma$ and $H\delta$ excited by condensed discharges for a pressure range from 48 to 250 mm of mercury was determined by photographing the spectra through a neutral wedge. In each case the broadening was symmetrical and the general form of the intensity curve was convex to the axis of wave-lengths, but the curves for each line gave evidences of structure characteristic of the line. A theoretical analysis yielded the following expression for the intensity I_λ at wave-length λ in the broadened line whose center was at λ_0: $\log I_\lambda = \log [a/(\lambda_0 - \lambda)] - b/(\lambda_0 - \lambda)$, where a and b are quantities which vary

with the line and with the pressure; b is small. This equation agrees with the general form of the experimental intensity curve. In the derivation the luminous atoms of the gas were considered to be subjected to electric fields caused by the charged particles of the gas, the electric fields being allotted according to a probability law. Because of the electric field, the line was resolved into the Stark components which were displaced from the parent line by an amount given by a general relation due to Sommerfeld. According to this theory the action of the condensed discharge in producing widening of the line is attributed to the high value of the current density through the gas in the early portion of the discharge, rising perhaps to hundreds of amperes/cm^2. The slight structure in the broadened line may be explained in terms of the observed intensities and positions of the components of the electrically resolved line.

5. Acoustic power of certain sound sources in absolute units. PAUL E. SABINE, Riverbank Laboratories, Geneva, Illinois.—From the duration of audible sound after the source has ceased in a room in which the absolute rate of dissipation is known, the rate of emission of sound in terms of the threshold of the observer can be determined. Measurement of the latter in absolute units was made with the thermophone by Kranz' method for the whole range of musical tones. The absolute powers of a number of sources at various pitches were thus measured. The approximate results are summarized below.

Source	Pitch Range d.v.	Acoustic Power ergs/sec.
Violoncello (bowed forte)	129– 650	1,000– 10
Violin (bowed forte)	129–1,300	600
Open Diapason Organ Pipes at wind pressure of .13 lb per in.^{2}	129–3,100	32,000–1,400
Fundamentals of 12 vowels intoned at speaking loudness by masculine voice	129	55

By modifying acoustic conditions in the room, it was possible to determine the power in the characteristic overtones of vowels of the first class. The results for these vowels intoned by a masculine voice are as follows:

Vowel	Characteristic Frecqueny	Acoustic Power
AH	870	225
AW	461	200
O	410	30
ÒO	326	10

6. The magnetic susceptibility of oxygen, hydrogen and helium. A. P. WILLS and G. HECTOR, Columbia University.—The chief object of the investigation was the determination of the magnetic susceptibility of hydrogen, on account of the importance of this quantity in atomic theory. The susceptibility of hydrogen is excessively small and that of helium still more so, therefore both are very difficult quantities to measure. For this reason preliminary experiments were made upon oxygen in an attempt to perfect the method before applving it to the investigation of hydrogen and helium. The principle of the method used was that of balancing the gas magnetically against an aqueous solution of nickel chloride. By varying the concentration of the solution it could be given a susceptibility approximately the same as that of the gas; then by varying the pressure of the gas or the temperature of the gas and solution, the two could be given the same susceptibility. A manometric balance of great sensitivity enabled the observer to tell when the susceptibilities of the gas and solution were the same. Observations at different temperatures and pressures furnished the data from

which the susceptibility of the gas could be calculated, the concentration of the solution being known; in these calculations the commonly accepted value for the susceptibility of water, -0.72×10^{-6}, was assumed. The following values were found for the volume susceptibility for a pressure of one atmosphere and a temperature of 20° C: for oxygen, 14.470×10^{-8}; for hydrogen, -1.70×10^{-10}; for helium, -0.80×10^{-10}. Further analysis of the experimental data may result in slight modifications of these numbers.

7. Dispersion of light by an electron gas. LEIGH PAGE, Yale University.—From the agreement in phase of the blue and yellow light curves of distant stars, Shapley has drawn the conclusion that the velocity of blue light through space cannot differ from that of yellow light by as much as one meter per second. Hence it seems of interest to investigate whether the laws of electro-magnetism lead to dispersion of light passing through streams of electrons or protons such as may be traversing interstellar space. The velocity q of light through a homogeneous electron gas is found to be

$$q = c(1 + ne^2\lambda^2/8\pi^2mc^2),$$

where n is the number of electrons per unit volume. From this expression an upper limit to n may be calculated. For electrons, the maximum permissible value of n in interstellar space is $(10)^{12}$ per cm^3, and for protons $(10)^{15}$ per cm^3.

8. The cause of ionization in the carbon arc. K. T. COMPTON, Princeton University.—Since the cathode drop in potential tends to adjust itself to the minimum value at which molecules of the surrounding gas may be ionized by collision, it seems safe to conclude that ionization near the outer boundary of the region of the cathode drop is due to collisions made with gas molecules by electrons emitted from the cathode. This accounts for most of the ionization in the arc. Beyond this region, and extending to the region of the anode drop, there is a small uniform potential gradient. We see, by Poisson's equation, that this must be a region of zero space charge, and therefore of some ionization to provide positive ions in sufficient number to neutralize the electronic space charge. An analysis of the conditions of ionic movement permits a calculation of the concentration of positive ions in this region. It is found that their number is entirely too large to be explained by impact ionization, but that thermal ionization, estimated by the method of Saha, is necessary and is probably an adequate explanation of ionization in this region of small uniform potential gradient.

9. Radiation potentials of atomic hydrogen. P. S. OLMSTEAD, American Telephone and Telegraph Company and the Western Electric Company, and K. T. COMPTON, Princeton University.—This investigation made use of several unique devices, which made it possible for the authors to determine accurately seven radiation potentials of atomic hydrogen. The tube used in the study contained a furnace by means of which the hydrogen was kept in the atomic form. This furnace was similar to one previously described (Compton, J. Opt. Soc. Amer. 8, 910, 1922). To eliminate effects from ionization, a set of platinum plates charged alternately positive and negative was placed between the source of radiation and the receiving electrode. An equipotential source of electrons consisting of a platinum point coated with the Wehnelt oxides was a feature of the apparatus. It was heated by the furnace and by thrusting it in various distances, emission of electrons from it could be changed. Potentials corresponding to the first six and convergence frequencies of the Lyman series were obtained in extremely close agreement with Bohr's theory. In no case did an observed break occur more than .05 volt from its predicted position. In the most favorable curves which were taken the observed breaks were within .01 volt for the first four and the convergence frequencies.

10. Velocity loss of cathode rays in matter. H. M. TERRILL, Columbia University.—The loss of velocity suffered by a beam of cathode rays on passing through a sheet of metal foil was measured by a modification of an apparatus used by Whiddington. A vacuum tube fitted with a Coolidge cathode was excited by high tension direct current. The electron stream of known velocity traversed the metal foil, and after deflection by a magnetic field, passed through a fixed slit and fell upon a collecting plate. The

magnetic field was calibrated by observations with the foil removed. After passing through the foil, the emerging beam was no longer homogeneous, and upon deflection in the magnetic field, was spread into a velocity spectrum. The intensities at various points of the velocity spectrum were measured by a gold leaf electroscope attached to the collecting plate. Distribution curves were obtained showing the maximum intensity to be toward the high velocity end. The velocity corresponding to maximum intensity was taken to be the effective velocity of the emerging beam. The results are in agreement with the law of J. J. Thomson, the difference between the fourth powers of the entering and emerging velocities being proportional to the thickness of the foil. The constants of proportionality obtained for various metals are as follows when multiplied by 10^{-43}: beryllium 0.94, aluminum 1.4, copper 4.5, silver 5.3, and gold 11.2. These constants are proportional to the densities of the metals.

11. A corpuscular quantum theory of the scattering of x-rays by light elements. G. E. M. JAUNCEY, Washington University.—A. H. Compton (American Physical Society, Dec. 1, 1922) has shown that the change of wave-length produced when x-rays are scattered can be explained by assuming that a quantum $h\nu$ is deflected when it comes near to an electron. The principles of the conservation of energy and momentum are applied to the system consisting of the quantum and the electron. This method is extended by the writer to explain the variation in the intensity with the angle of scattering. If a law of force between the quantum and the electron is assumed such that Thomson's scattering formula holds when the momentum $h\nu/c$ of the quantum is small (the recoil velocity of the electron then being negligible), it is found that when $h\nu/c$ becomes large so that the electron recoils with considerable velocity, the scattering formula becomes modified, so that although the scattered intensity remains the same as that of Thomson's theory for zero scattering angle, yet at other angles the scattering curve drops below Thomson's curve. The theoretical curve agrees very well with A. H. Compton's experimental results on the scattering of gamma rays from iron. On this theory, since it is only when a quantum hits an electron that the quantum is scattered, it is possible to determine the effective *radius of the electron* from the experimental value of the scattering. This radius is found to be 4.57×10^{-13} cm.

12. A note on the scattering of x-rays by liquid benzene, mesitylene and octane, and by diamond splints. C. W. HEWLETT, General Electric Research Laboratory.—The method of procedure was that described by the author (Phys. Rev. **20**, 688, 1922), except for a narrower slit in the ionization chamber, giving better resolution and a more accurate idea of the distribution of the scattered radiation. The scattering curve for octane shows three maxima due respectively to general radiation, and first and second order reflection for the K radiation of Mo. Benzene and mesitylene curves show the general radiation, but it is not resolved from the K radiation. Mesitylene shows two maxima for the K radiation, giving spacings for the atomic planes of 6.2 A and 4.1 A. The effect of the general radiation is a function of the amount of methyl bromide in the ionization chamber. Comparison of curves for diamond splints at room temperature, with curves taken at 300° C, shows slight decrease of intensity of some of the maxima, and some small displacements at large angles at the higher temperature. The accuracy of the work was not sufficient to justify any quantitative conclusions from the shape of the scattering curves. Experiments in which the total scattered radiation was integrated gave the same value at room temperature and at 300° C to within 1 per cent. This work was done during the summer of 1922 at the State University of Iowa.

13. The effect of tube-lead capacitance on the efficiency of x-ray apparatus in which inverse current is suppressed by kenetrons. HARRY CLARK, The Rockefeller Institute for Medical Research.—An arrangement is described in which, for convenience, the tube is placed at some distance from the rest of the apparatus. Because of the considerable capacitance of the tube-leads, a kenetron is placed at each end of the high-voltage transformer which is grounded in the middle. Since the system is nearly symmetrical,

the equivalent capacitance of either lead under normal condition of operation may be taken as $i/E\omega$, where i is the charging current for a lead read on an a.c. milliammeter when the kenetrons are short-circuited, the tube filament cold, and the transformer is operating at voltage E and frequency ω. When working normally at any voltage V (r.m.s.), the instantaneous voltage on the tube is the same as that of the transformer during the useful half of the cycle, provided that the tube-current, j, read on a d.c. meter is equal to or greater than $\frac{1}{2}i_r\sqrt{2}$. In the arrangement described, 4 milliamperes satisfies this condition at 70 Kv and 60 cycles. As j decreases from the value $\frac{1}{2}i_r\sqrt{2}$, the hard-ray efficiency decreases and passes through a minimum when j becomes equal, approximately, to $.2i_r\sqrt{2}$. The theory is supported by ionization measurements. The design of similar apparatus for high voltage is discussed.

14. Absorption of short x-rays by water and carbon. F. K. RICHTMYER, Cornell University.—Previous measurements by the writer, on the absorption of x-rays by water indicate that water does not obey the linear relation between the mass absorption coefficient and the cube of the wave-length in the region $\lambda = .1$ A to $\lambda = .5$ A. Because of the high (theoretical) value of the mass scattering coefficient of hydrogen, as well as of the importance of accurate data on the absorption by water, some very careful measurements have been made with a new and improved set-up. These measurements confirm, both qualitatively and quantitatively, the previous measurements of the writer, and are not in agreement with those reported by Hewlett. Preliminary data on carbon seem to show that carbon agrees with the λ-cube law to much shorter wave-lengths than water.

15. The variation of contact potential difference with temperature in potassium, measured photo-electrically. HERBERT E. IVES, American Telephone and Telegraph Company and the Western Electric Company.—Two obvious photo-electric methods of detecting a change in the work function of a surface emitting photo-electrons are by the shift of the long wave-limit of the photo-electric emission, and by the change in value of the saturation voltage. However, these methods are not capable of great accuracy, and in the study of the variation of photo-electric current with temperature (Physical Review, **20**, 102, July, 1922) have not been found to give unambiguous results. A third method is available when the receiving electrode is also photo-sensitive, as is usually the case in working with alkali metals owing to the condensation of metal vapor on all parts of the photo-electric cell. This method rests on the experimental fact that no change in the cathode alters the potential necessary to be applied to it to stop photo-electrons excited by a definite frequency. It can be shown that what must happen in the case of a change of work function is a shift of the stopping potential and saturation voltage for the other, unchanged electrode; that is, a bodily shift of its voltage-current curve along the voltage axis. This shift is equal to the variation in contact potential difference, provided the photo-electric emission from the altered electrode is negligible. If it is not negligible, the shift is detected by a movement of the point where the complete voltage-current curve cuts the voltage axis. Such a displacement of the voltage-current curve is shown when one of a pair of potassium electrodes is cooled to liquid air temperature.

16. The effect of temperature on the dark current produced in a platinum-rhodamine B-platinum photo-active cell. FREDERICK A. MAY, Cornell University.—(C. C. Murdock, Phys. Rev. **17**, p. 626, 1921; R. Y. Jenkins, Phys. Rev. **18**, p. 402, 1921.) A U-shaped cell was arranged so that either electrode could be heated by means of an oil bath, the temperature difference being measured with a thermo-electric thermometer. The dark current was generally found to be independent of the resistance of the external circuit, as shown by Murdock. In some experiments this effect was partly masked by the presence of a chemical electromotive force, but this could be practically eliminated by introducing sufficient resistance into the circuit. Heating either electrode made it more positive, so that heating the negative electrode caused a reduction of the dark

current, which could thus be made to reverse its direction. Soon after passing through zero, a slight increase in temperature caused a large and sudden increase in the current, this increase sometimes exceeding ten times the original dark current. Further increase in temperature caused only a slight increase in the current. As the electrode cooled the current returned more slowly to its original value. The effect is thought to be due to some semi-permanent change in the nature of the electrodes, resulting from electrolytic action.

17. The contact electricity of solid dielectrics. HAROLD F. RICHARDS, Princeton University.—The fact that the metals are distributed among the dielectrics in the tribo-electric series suggests the possibility of establishing a contact theory which will include solid dielectrics. This conclusion has been tested by measuring the electrification produced by wringing two optically flat surfaces together. The results for different pairs of the substances quartz, fluorite, flint glass, crown glass, and steel satisfied the equation $Q_1 = 4.43(K_1 - K_2)$. Here Q_1 is the charge per cm² in e.s.u. of the insulator whose dielectric constant is K_1; and K_2 is the dielectric constant of the other specimen. The results for contact of steel and an insulator fit the equation if $K = 3.1$ be assigned to steel. Within wide limits the charge was independent of the amount of friction and the duration of contact; and was not diminished when the residual air between the contact surfaces was ionized with x-rays. These results prove that electrification by friction is a contact effect.

The four insulators, which were positive to steel in the contact series, became negative when *colliding with steel.* This may be explained by assuming that collision produces not only the true contact effect, but also a transfer of electrons from the metal to the dielectric due to the inertia of the mobile electrons.

18. Duration of resonance radiation in mercury vapor. HAROLD W. WEBB, Columbia University.—Vapor at 50° C in a quartz tube with a central hot cathode surrounded by a wire mesh was excited by electron impacts of less than 10 volts. The radiation passed out to a photo-electric system, a nickel cylinder and two gauzes, which gave negligible current except when the gauze was positive. Alternating voltages applied to the mesh gave excitation only during the positive half-cycle. Voltages of the same frequency were applied to the photo-electric gauze first in phase and then out of phase, so that it acted either during the period of electron impact or during the next half-cycle. With low frequencies no photo-electric current was observed when the voltages were out of phase. With frequencies increasing from 10^5 to 5×10^5 the out-of-phase current increased due to the persistence of the radiation, while the in-phase current decreased. Assuming the decay of radiation to be represented by e^{-kt}, k was computed to be 10^6 sec^{-1}. The exact law of decay and its dependence upon the depth of the vapor has not yet been determined.

19. The initial change in the e.m.f. between a metal plate and a solution after being suddenly brought into contact. R. D. KLEEMAN and R. H. BENNETT, Union College.—Two metal plates of the same material were connected by two wires with the terminals of a galvanometer, one of the plates being kept immersed in a solution and the other initially suspended above the solution by an appropriate arrangement. The suspended plate was next suddenly plunged into the solution, and the galvanometer deflections observed. If the difference of potential between the solution and the plate which is suddenly immersed is formed gradually, the direction of the initial current through the galvanometer could be predicted from a knowledge of the difference of potential between plate and solution under conditions of equilibrium. It was found that the e.m.f. is not formed equally rapidly for different combinations of metals and solutions, and possesses a marked inertia effect, shown by a temporary increase beyond that corresponding to equilibrium; in fact, it appeared to oscillate about the equilibrium e.m.f. The initial increase in e.m.f. in some cases was so rapid that the initial current was not of sufficient duration to be observed, and at first there appeared a strong temporary

current in the opposite direction. The results may be well explained by a theory of the voltaic e.m.f. given by the writer (Physical Review, **20**, p. 174, 1922).

20. Diffraction caustics due to apertures. G. G. BECKNELL and JOHN COULSON, University of Pittsburgh.—Diffraction figures produced by circular, elliptical, and other shaped apertures illuminated by a point source are found to be identical with those produced by any object completely filling the aperture; that is, in every case the predominant figure is the evolute of the boundary of the geometrical shadow. Portions of the evolutes lying either outside or within the illuminated areas of apertures, as well as the diffraction normals which traverse both the shadows and illuminated areas, are shown photographically. To the latter may be ascribed the formation of the evolutes and their corresponding families of secondary curves. Quite distinct from the spot and rings due to a circular aperture, with the color transitions when the distance of the point of observation is changed, is a superposed spot identical with the Arago spot for a disk, which remains white under all conditions. This has been observed and photographed. A similar spot exists for any aperture having a circular projection in the focal plane. A heart-shaped aperture (involute of a circle) furnishes within its circular evolute an area which is entirely free from diffracted light from the boundary of the aperture. Such apertures may be used advantageously in the construction of optical instruments.

21. Theory of conductivity and specific heat measurement at high temperature. MAYO D. HERSEY, United States Bureau of Mines, Pittsburgh.—This paper gives detailed working formulas for determining conductivity and specific heat by the principle of physical similarity (theory of dimensions), a method which simplifies the experimental requirements by dispensing with uniform or instantaneous temperatures, simple geometrical shapes, guard rings and the like. The method has been developed in connection with the investigation of refractory materials at high temperature at the Bureau of Mines Physical Laboratory, but may be of equal interest in other classes of work. It is a particular application of the "General Method for Determining Properties of Matter," previously published in mathematical form only (J. Wash. Acad. Sci. **12**, 167–172, 1922).

22. Sensitivity of the ear as a function of pitch. FREDERICK W. KRANZ, Riverbank Laboratories.—A vacuum tube oscillator furnished electrical currents which were used in connection with a thermophone to produce suitable sounds, the pitch being controlled by a continuously variable inductance so that continuous ranges of pitch were available. The determination of the audible portions of a frequency range for various levels of loudness allowed the curve of sensitivity as a function of frequency to be accurately determined with negligible assumptions as to continuity. Absolute values of the necessary intensity for audition were obtained. The method is considered to be especially well suited to the study of tonal gaps. The results showed very abrupt changes in sensitivity over the frequency range in many cases, the form of the curves varying greatly for different ears, even for the two ears of a single individual. Changes as great as a factor of 1000 in necessary intensity for audition were found with a change of pitch of a semi-tone. In most cases, the irregularities were found to occur above a frequency of 1500 p.p.s. An acceptable theory of the mechanism of audition should include an explanation of these abrupt changes.

23. The reduction of sound transmitted by plaster walls as a function of thickness. PAUL E. SABINE, Riverbank Laboratories.—Plaster walls 8 x 6 feet, ranging in thickness from $1\frac{1}{2}$ to $4\frac{1}{2}$ inches, were built between two rooms structurally separated from each other, so that all passage of sound between the two was by way of the walls undergoing test. The relative intensities of sound from a source in the larger room were determined by measuring the duration in the two rooms, of audibility of the residual sound after the source had ceased. The relative intensities in such a case are given by the equation $\log (I_1/I_2) = A(t_1 - t_2)$, where A is the absolute rate of decrease of intensity with time in the room containing the source, and is known from a preliminary calibration

of this room. The tests included 23 tones covering the tonal range from 129 to 4139 p.p.s. The curve of the reduction of intensity as a function of pitch for a wall of given thickness is not smooth at the low pitch end where resonance in the wall itself plays an important part. In general, the reduction of intensity increases with rising pitch, an abrupt increase being noted at a certain pitch. This region of abrupt change moves downward in the scale as the thickness of the wall is increased. The frequency reduction curves for different thicknesses are generally similar. The logarithms of the reduction factors averaged for all tones for thicknesses of 3/2, 5/2, 7/2 and 9/2 inches are 2.53, 3.24, 3.60 and 3.80, respectively, the corresponding reduction factors being 340, 1740, 4000 and 6300. The straight line relation between thickness and logarithmic reduction which was found to hold for felt-like materials for all frequencies does not hold for plaster walls, the increase in the logarithm per unit increase in thickness decreasing with the thickness.

24. The quantum theory and wave propagation. W. M. STEMPEL, Stevens Institute of Technology.—By a slight modification of Planck's idea it is possible to derive his general radiation formula without having to assign so strange a behavior to the oscillator as Planck had to do. The Equipartition Theory and the theory of the Continuity of Physical Effects are kept intact. When energy is radiated in electro-magnetic waves, it is easily seen that the energy arrives in puffs. An oscillator, whether vibrating back and forth or in a nearly circular orbit, cannot radiate continuously for all parts of the path. Wave propagation is the outer manifestation of this fact. Looked at from this point of view, the energy radiated during each half oscillation need not be the same. Apply Planck's reasoning to a class of oscillators which for the moment are radiating equal amounts of energy per half vibration and one arrives at Planck's formula. The effect summed up for all classes of oscillators must bring one to the same general black body radiation curve. Planck's constant is thus not the energy associated with a single emission but is a statistical average, much the same sort of thing as the mean velocity in the kinetic theory of gases, and as in that case is a function of the state of agitation of the oscillator.